Almost everywhere convergence of Fourier series

Basic Notions Seminar

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1 Fourier series

The functions $(e_n)_{n\in\mathbf{Z}}$ defined by $e_n(t) = e^{2\pi i n t}$ form an orthonormal basis of the Hilbert space $L^2([0,1))$. Thus we have for $f \in L^2([0,1))$,

$$f = \sum_{n\in\mathbf{Z}} \langle f, e_n\rangle e_n \tag{1}$$

By this equation we mean only that the sum on the right hand side converges to f in the L^2 norm. That is, denoting the partial sums by

$$S_N f = \sum_{n=-N}^{N} \langle f, e_n\rangle e_n$$

equation (1) just means

$$\lim_{N\to\infty} \|f - S_N f\|_2 = 0 \tag{2}$$

The coefficients $\langle f, e_n\rangle = \int_0^1 f(t)e^{-2\pi i n t}dt$ of f are also called *Fourier coefficients* of f and denoted $\widehat{f}(n)$. The above basis expansion is the *Fourier series* of f.

It is an interesting question whether the partial sums of the Fourier series $S_N f(t)$ also converge at a given point $t \in [0,1)$ to the correpsonding value $f(t)$.

If f is, say, differentiable at t it is not hard to show that $S_N f(t)$ does converge to $f(t)$. But if f is only continuous at t, it is not clear whether the sequence $S_N f(t)$ even converges[1]. Using the principle of uniform boundedness, one can construct continuous functions whose Fourier series diverge at a given point.

With some more effort one can show that for any given set $E \subset [0,1)$ of Lebesgue measure zero, it is possible to construct a continuous function which diverges on E.

It was a long standing conjecture by Luzin (1915) that the Fourier series of a continuous function converges almost everywhere. In the 1920s, Kolmogoroff gave an example of an L^1 function whose Fourier series diverges everywhere. The conjecture was settled by Lennart Carleson in 1965 [1] who proved the following even stronger result.

[1]But by means of Cesáro summation we at least know that, *if* it converges, the limit has to be $f(t)$.

Theorem 1 (Carleson). *For $f \in L^2([0,1))$, $S_N f(t)$ converges to $f(t)$ for almost every $t \in [0,1)$ as $N \to \infty$.*

This was extended to L^p for $1 < p < 2$ by R. Hunt (1968).

2 The Carleson operator

There are essentially three different approaches to proving Carleson's theorem. Carleson's original paper [1] is known to be notoriously difficult to read and understand. It introduces a new technique which has since developed into part of what is known today as time-frequency analysis.
All proofs start out in a way that is very typical for pointwise convergence questions. They bound a corresponding maximal operator which can be thought of as measuring the error when trying to approximate f by smooth functions, for which pointwise convergence is known. This operator is called the *Carleson operator* and given by

$$\mathcal{C}f(t) = \sup_{N \in \mathbf{Z}} \left| \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n t} \right|$$

It is important to notice that this operator is *not* linear, but only sublinear.
The original proof given by Carleson uses a sophisticated decomposition of the function f. In 1973, Charles Fefferman [2] gave a simpler proof of Carleson's theorem focusing on decomposing the operator. The approach by Michael Lacey and Christoph Thiele [3] in 2000 unifies both previous proofs in that it works on both the operator and the function.
The essence of time-frequency techniques is to break up a problem in terms of its symmetries. In the second half of the talk, we will try to sketch some of the major steps in the Lacey-Thiele approach. The techniques can be adapted to tackle many other problems in harmonic analysis.

References

[1] Lennart Carleson. On convergence and growth of partial sums of Fourier series. *Acta Math.*, 116:135–157, 1966.

[2] Charles Fefferman. Pointwise convergence of Fourier series. *Ann. of Math. (2)*, 98:551–571, 1973.

[3] Michael Lacey and Christoph Thiele. A proof of boundedness of the Carleson operator. *Math. Res. Lett.*, 7(4):361–370, 2000.