Kleine AG
Elliptic Surfaces

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Organization:
Heinrich Hartmann (hartmann@math.uni-bonn.de),
Holger Partsch (partsch@math.uni-duesseldorf.de)

1. Introduction

An *elliptic surface* is a smooth surface X that admits a connected, proper morphism $f : X \to C$ from onto a smooth curve C such that a general fiber is an elliptic curve.

Elliptic surfaces form an interesting class of varieties for various reasons.

There are plenty of examples: Every surface of Kodaira dimension 1 is elliptic. A $K3$ surface, with a divisor $D \neq 0$ satisfying $D^2 = 0$ admits an elliptic fibration. Also elliptic curves over $Spec(\mathbb{Z})$ are examples of elliptic surfaces.

An elliptic surfaces (with a section) has a convenient representation as Weierstraß-model, that is a birational map to a surface in $\mathbb{P}^2 \times C$, given by an explicit equation. As a first application we will see, how to classify the singular fibers of f. Moreover it is possible to analyze the topology of X in a very direct way. One can compute monodromies, homology classes, and their intersections (e.g. [NS94]).

The irreducible components of the (singular) fibers of f provide us with many interesting divisors of X. A theorem of Shioda and Tate states that the remaining part of $NS(X)$ is controlled by the Mordell–Weil group of sections of f. In some case this is enough to compute the whole Neron–Severi group of X (e.g. [Dol96], p.28).

Hence elliptic surfaces are not only an interesting topic with rich theory but also a useful technique for the real life of an algebraic geometer. In this *Kleine AG* we will try to learn some of this techniques mentioned above. In the last talk we will touch upon a more recent result by Brigeland [Bri98], which gives derived equivalences between an elliptic surface and some associated moduli spaces (like the jacobian). For this we will have to assume familiarity with derived categories and Fourier–Mukai transformations. The remaining talks will get by with standard algebraic geometry knowledge. As a preparation for the workshop we encourage the participants to read the excellent survey in the Encyclopedia of Mathematical Sciences by Shafarevich et. al. [IS89].

2. The Program

1. Lecture (45 min.): Weierstraß-Models and Degeneration. ([MS72] Chapter 3, [SS09] Section 4)
Given an elliptic curve (E, O), one can use the very ample line bundle $\mathrm{O}_E(3O)$ to obtain an embedding of E into $\mathbb{P}^2$. This embedding is given by a certain Weierstraß-equation for E. This theory generalizes in a natural way to elliptic surfaces with section. This is done in [MS72] Chapter 3, after definition 1. In the reference, the fibration is assumed to be smooth, but it is enough to assume that the fibration is flat and the generic fiber is elliptic. Given $X \to C$ with section $s : C \to X$, we want to see how we can find a Weierstraß-Model of X given by a relative Weierstraß equation inside some projective space $\mathbb{P}(\mathrm{F})$ where F is some rank three bundle over C ([MS72] Theorem 1').
The Weierstraß-Model is always normal, but in general singular. We are now going to study what kind of singularities appear and how to resolve them. Our reference will be [SS09, Sections 4.1 - 4.3]. This question is of local nature, so we take the base to be the spectrum of a discrete valuation ring. If the Weierstraß-Model is minimal, we are dealing with rational

double point singularities. Those are classified by the Dynkin diagrams of their resolution graphs. It would be nice to see some of those. We also want to understand the difference between additive and multiplicative reduction, and see an example for the resolution of a bad fiber by blowups, i.e an working example of the Tate-Algorithm.

2. Lecture (45 min.): Jacobians and classification results over $\mathbb{C}$. ([BPV84], V.9-11.) In this section we work over the complex numbers. So an elliptic fibration is for us a proper, connected, holomorphic map $f : X \to C$ form a smooth surface onto a smooth curve, such that the general fiber is non-singular elliptic.

In this setting we can construct the relative jacobian as $Jac(f) := R^1 f_* \mathcal{O}_X / R^1 f_* \mathbb{Z}_X$. This works literally only over the regular part. Explain how to use a stable reduction, and the explicit description of the monodromy around a stable fiber to extend the jacobian to the singular fibers. You will have to recall this techniques form the earlier chapters.

If f admits a section the jacobian is birational to X. Proof this (easy).

We now want to see how to the jacobian can be used to classify elliptic surfaces (without multiple fibers). Introduce the *homological invariant* $R : \pi_1(C_{reg}) \to SL(2, \mathbb{Z})$, and the *functional invariant* $J : C_{reg} \to \mathbb{C} = SL(2, \mathbb{Z}) \backslash \mathbb{H}$.

For given invariants J and R there is a unique elliptic fibration f admitting a section. Moreover the set $F(J, R)$ of elliptic fibrations with this invariants is isomorphic to $H^1(S, Jac(f))$, the first homology group of the sheaf of holomorphic sections in the jacobian. We will not have time to proof this in detail, but give at least the main ideas.

3. Lecture (45 min.): Relations to the theory of surfaces. ([Bădo1] Section 7,8) In the theory of surfaces, those fibered in elliptic curves, play an important role. Most prominently for every surface of Kodaira dimension one, the Stein factorization of the canonical map gives a elliptic fibration.
All references in this section are taken from [Bădo1]. We begin with the definition of a curve of canonical type (Definition 7.7), and see why fibers of elliptic fibrations are examples. Next state the partial converse Theorem 7.11 and Corollary 7.13.
Now we shift our attention to the canonical sheaf of a elliptic surface: we want to understand the canonical bundle formula (Theorem 7.15) and see a sketch of the proof. As an application we have Theorem 8.1 and Remark 8.3. To end with an concrete example, give an overview about hyperelliptic surfaces. (Theorem 8.10 and Section 10.24).

4. Lecture (45 min.): The Néron-Severi lattice of an elliptic surface. [SS09, Chapter 6] It is a deep theorem of arithmetics, that the Mordell-Weil group of an elliptic curve over a number field is finitely generated. We are going to show the same for elliptic surfaces with sections, by using geometric arguments, especially the Néron-Severi lattice.
This talk covers chapter 6 of [SS09]. We make the assumption that our fibrations are not of product type, which is implied by the existence of at least one singular fiber.
We start out with the definition of the Mordell-Weil group and Néron-Severi group. Our aim is to proof Theorem 6.1. The argumentation goes as follows: Showing that numerical and algebraic equivalence coincide, we get at once that $NS(X)$ is finitely generated. Now we realize the Mordell-Weil group as a quotient of $NS(X)$ by the trivial lattice (6.7), and we are done.
The formula $12\chi(X) = e(X) > 0$ from (6.11) and the expression of $e(X)$ by the fiber types (6.10) are of independent interest. The final result is Corollary 6.13, showing that $\rho(X)$ can be computed by counting fiber components and adding the Mordell-Weil rank.
If time allows it, one can do the example (6.12).

5. Lecture (45 min.): Relative moduli spaces and Fourier–Mukai transforms. Atiyah showed, that the the components of the moduli space of stable sheaves of coprime rank $r > 0$ and degree d on an elliptic curve C, are again elliptic curves isomorphic to C.

Given an elliptic surface $f : X \to C$ over the complex numbers, there are relative versions of these moduli spaces which give us elliptic surfaces $Y = J_X(r, d) \to C$, for any pair r, d as above. Note that $Jac(f) = J_X(1, 0)$.

Moreover there are tautological sheaves P on $X \times Y$ supported on $X \times_C Y$ which induce Fourier–Mukai transforms

$$\Phi^P : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$$

that turn out to be equivalences of categories. This results were proved by Bridgeland in his PHD thesis [Bri98].

In the talk we want to see (a sketch of) the construction of $J_X(r, d)$ as relative moduli space of stable 1-dimensional sheaves on X. This is explained in the first few lines of Section 4.2, however no details are given so one need to look at the literature to see what $M(X/C)$ really is ([Sim94] or [HL96]) and how to obtain a universal sheaf ([Muk87], A.6). It follows then easily, that $J_X(r, d)$ is connected and an elliptic surface.

It remains to show, that the induced Fourier–Mukai transform Φ^P is indeed an equivalence. We use the Bondal–Orlov criterion ([Bri98], Theorem 2.1) to show fully-faithfulness. As this only involves the fibers P_y which are stable sheaves on an elliptic curves this is not hard (see Section 4.2). The essential surjectivity is easy nowadays. We can directly apply the criterion Bridgland gave later in 1999 (see [Huy06] 7.11) to shortcut the arguments in the paper. Maybe it is possible to indicate a proof of the criterion?

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