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Relative Trace Formulae in Analytic Number Theory

Set 2: Analytic applications of the Kuznetsov formula

In the beginning we will state two versions of the Kuznetsov formula that can be used as black boxes.[1] After stating the exercises there is a list of useful facts that might be helpful solving them.

Basic notation: Let $\mathbb{H} = \{z = x + iy \colon x \in \mathbb{R},\, y > 0\}$ be the upper half plane equipped with the measure $dz = \frac{dxdy}{y^2}$ and the hyperbolic Laplacian $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2})$. The Hecke operators are defined by

$$[T_n f](z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n,\\ d>0}} \sum_{b \bmod d} f(\frac{az+b}{d}) \text{ for } n \geq 1 \text{ and } [T_{-1}f](z) = f(-\overline{z}).$$

We write $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We have the action

$$\gamma.z = \frac{az+b}{cz+d} \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } z \in \mathbb{H}.$$

Let $\mathcal{F} = \{z = x + iy \colon |x| \leq \frac{1}{2},\, |z| \geq 1\}$ be the standard fundamental domain for $\Gamma\backslash\mathbb{H}$. The space $L^2(\Gamma\backslash\mathbb{H})$ of Γ-invariant functions on $\mathbb{H}$ that are square integrable on $\mathcal{F}$ has a spectral expansion featuring the constant function ϕ_0 with Δ-eigenvalue $\lambda_0 = 0$, so called Maaß cusp forms and Eisenstein series.

Maaß Forms: A Hecke-Maaß cusp form ϕ is a square integrable eigenfunction of Δ that is also an eigenfunction of all Hecke operators T_n with $n \in \mathbb{Z}$. We sort the corresponding Laplace eigenvalues by size and numerate them: $0 < \lambda_1 \leq \lambda_2 \leq \ldots$. The corresponding Hecke-Maaß cusp forms are denoted by $\phi_1, \phi_2, \ldots$. Note that $\frac{1}{4} < \lambda_1$ and $\lambda_j \sim 12j$ as $j \to \infty$. These are already non-trivial facts. We normalize our Maaß forms by

$$\langle \phi_i, \phi_j \rangle = \int_{\mathcal{F}} \phi_i(z)\overline{\phi_j(z)}dz = \delta_{i=j} \text{ for } i, j \in \mathbb{Z}_{\geq 0}.$$

One has the Fourier expansion

$$\phi_j(z) = \sqrt{y} \sum_{n \neq 0} \rho_j(n) K_{it_j}(2\pi|n|y)e(nx) \text{ where } t_j = \sqrt{\lambda_j - \frac{1}{4}}.$$

If $\lambda_j(n)$ denotes the eigenvalue of the n'th Hecke operator (i.e. $T_n\phi_j = \lambda_j(n)\phi_j$), then one can compute that

$$\rho_j(n) = \rho_j(1)\lambda_j(n) \text{ and } \rho_j(-n) = \epsilon_j\rho_j(n) \text{ with } \epsilon_j = \lambda_j(-1) \in \{\pm 1\}.$$

Finally we associate the L-function

$$L(s, \phi_j) = \sum_{n \geq 1} \lambda_j(n)n^{-s} = \prod_p (1 - \lambda_j(p)p^{-s} + p^{-2s})^{-1}.$$

[1] Some of the notation differs from the one used in the lecture series. Sorry for the inconvenience.

Eisenstein Series: We define the Eisenstein series by

$$E(z,s) = \frac{1}{2}\sum_{\Gamma_\infty\backslash\Gamma} \Im(\gamma.z)^s \text{ where } \Gamma_\infty = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Z} \right\}.$$

For $\Re(s) > 1$ the series converges and defines a non square integrable eigenfunction of Δ with eigenvalue $\lambda = s(1-s)$. The Eisenstein series features a meromorphic continuation to all $s \in \mathbb{C}$. Further we have

$$T_n E(\cdot, \frac{1}{2}+it) = n^{-it}\sigma_{2it}(n)E(\cdot, \frac{1}{2}+it) \text{ for } \sigma_s(n) = \sum_{d|n} d^s.$$

Of course Eisenstein series feature a Fourier expansion (with constant term) and one can form a Dirichlet series using their Hecke eigenvalues.

The forward Kuznetsov Formula: For $m,n \in \mathbb{N}$ and $r \in \mathbb{R}$ we have

$$\pi\sum_{j=1}^{\infty} A(r,t_j)\overline{\rho_j(m)}\rho_j(n) + \int_{-\infty}^{\infty} A(r,t)\left(\frac{m}{n}\right)^{it}\sigma_{2it}(n)\sigma_{-2it}(m)\frac{\cosh(\pi t)}{|\zeta(1+2ir)|^2}dt$$
$$= \frac{r}{2\pi^2}\delta_{m=n} + \frac{r}{\cosh(\pi r)}\sum_{c=1}^{\infty}\frac{S(n,m;c)}{c}\cdot\frac{4\pi\sqrt{mn}}{c}\cdot\int_{4\pi\sqrt{mn}/c}^{\infty}[J_{2ir}(u)+J_{-2ir}(u)]\frac{du}{u},$$

where

$$A(r,t) = \frac{\sinh(\pi r)}{\cosh(\pi r)^2 + \sinh(\pi t)^2}$$

and

$$S(n,m;c) = \sum_{\substack{d \bmod c,\\ (c,d)=1}} e\left(\frac{nd+m\overline{d}}{c}\right).$$

The backward Kuznetsov Formula: Let $f \in \mathcal{C}_c^2(\mathbb{R}_{>0})$. Then for $m,n \in \mathbb{N}$ we have

$$\sum_{c=1}^{\infty}\frac{S(n,-m;c)}{c}f\left(\frac{4\pi\sqrt{mn}}{c}\right)$$
$$= 4\sum_{j=1}^{\infty}\rho_j(n)\rho_j(m)\tilde{f}(t_j) + \int_{-\infty}^{\infty}(nm)^{it}\sigma_{2it}(n)\sigma_{2it}(m)\frac{\tilde{f}(t)}{|\Gamma(\frac{1}{2}+it)\zeta(\frac{1}{2}+2it)|^2}dt$$

where

$$\tilde{f}(t) = \int_0^{\infty} K_{2it}(x)f(x)\frac{dx}{x}. \tag{1}$$

Exercise 2.1: Prove the following estimate: For large parameter T, N and any sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers we have

$$\sum_{T/2\leq t_j\leq T} \frac{1}{\cosh(\pi t_j)} \left| \sum_{N/2\leq n\leq N} a_n\rho_j(n) \right|^2 \ll (T^2+N)N^\epsilon \sum_{N/2\leq n\leq N} |a_n|^2. \tag{2}$$

a) For $N/2 < N_1 < N$ and $1 \leq |\theta| < 3$ define

$$B(c,N) = \sum_{N_1<n,m\leq N} \overline{b_m} b_n S(n,m;c) e(\theta\frac{\sqrt{nm}}{c}).$$

Show that

$$B(c,N) \ll c^{\frac{1}{2}+\epsilon} N \sum |b_n|^2 \text{ for all } c \geq 1, \tag{3}$$
$$B(c,N) \ll (c+N)N^\epsilon \sum |b_n|^2 \text{ for } c > N^{1-\epsilon} \text{ and} \tag{4}$$
$$B(c,N) \ll c^{\frac{1}{2}} N^{\frac{1}{2}+\epsilon} \sum |b_n|^2 \text{ for } c \leq N^{1-\epsilon}. \tag{5}$$

b) Choose a suitable function $\varphi(x)$ so that for $t \in [T/2, T]$ the bound

$$\int_0^\infty \varphi(x) A(x,t) dx > \cosh(\pi t)^{-1}$$

holds. Conclude that

$$\pi \sum_{T/2\leq t_j\leq T} \frac{1}{\cosh(\pi t_j)} \left| \sum_{N/2\leq n\leq N} a_n\rho_j(n) \right|^2$$
$$\leq \frac{1}{2\pi^2}\left(\int_0^\infty t\varphi(t)dt\right) \sum_n |a_n|^2 + \sum_{c=1}^\infty \frac{4\pi^2}{c^2} \sum_{m,n} \overline{a_m} a_n \sqrt{mn} S(n,m;c) \Phi(\frac{4\pi\sqrt{mn}}{c})$$

where

$$\Phi(x) = \int_0^\infty \frac{t\varphi(t)}{\cosh(\pi t)} \int_x^\infty (J_{2it}(u) + J_{-2it}(u)) \frac{du}{u} dt.$$

c) Show that $\Phi(x) = \Delta(x) + O(TN^{-2})$ where

$$\Delta(x) = \int\int K(t,z) \sin(x\cosh(z)) dtdz$$

for some $K\colon \mathbb{R}^2 \to \mathbb{C}$ with $\|L\|_{L^1} \ll \frac{N^\epsilon}{T}$. Furthermore, if $x > T^2$, then we have

$$\Delta(x) = \frac{1}{x}\int\int L(t,z)\cos(x\cosh(z)) dtdz$$

for some $L\colon \mathbb{R}^2 \to \mathbb{C}$ with $\|L\|_{L^1} \ll N^\epsilon T$.

d) Conclude the proof by combining the estimates obtained above.

Exercise 2.2: Prove the following estimate: For large parameter $L \geq 1$, $1 \leq N \ll K$ and any sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ of complex numbers we have

$$\sum_{T/2<t_j\leq T} \frac{1}{\cosh(\pi t_j)} \left| \sum_{N/2<n\leq N} a_n \rho_j(n) \right|^2 \left| \sum_{L/2<l\leq L} b_l \cdot l^{it_j} \right|^2 \ll T^{1+\epsilon}(T+L) \left(\sum_{n\leq N} |a_n|^2 \right) \left(\sum_{l\leq L} |b_l|^2 \right).$$

a) Use the function φ from Exercise 2.1, (2) and the forward Kuznetsov formula to show that

$$\pi \sum_{T/2<\kappa_j\leq T} \frac{1}{\cosh(\pi t_j)} \left| \sum_{N/2<n\leq N} a_n \rho_j(n) \right|^2 \left| \sum_{L/2<l\leq L} b_l \cdot l^{it_j} \right|^2$$
$$\leq S_1(T,L,N) + S_2(T,L,N) + O(TL\|a_n\|_2^2 \|b_l\|_2^2),$$

where

$$S_1(T,L,N) = \frac{1}{2\pi^2} \int_{\mathbb{R}} t\varphi(t) \sum_{l_1,l_2} \frac{b_{l_1}\overline{b_{l_2}}}{\alpha(l_1/l_2)} \left(\frac{l_1}{l_2}\right)^{it} dt \|a_n\|^2 \text{ and}$$

$$S_2(T,L,N) = \sum_{l_1,l_2} \frac{b_{l_1}\overline{b_{l_2}}}{\alpha(l_1/l_2)} \sum_{c=1}^{\infty} \frac{4\pi}{c^2} \sum_{m,n} \overline{a_m} a_n \sqrt{mn} S(n,m;c) \Phi\left(\frac{4\pi\sqrt{mn}}{c}, \frac{l_1}{l_2}\right) \text{ with}$$

$$\Phi(x,y) = \int_{\mathbb{R}} y^{it} \frac{t\varphi(t)}{\cosh(\pi t)} \int_x^{\infty} (J_{2it}(u) + J_{-2it}(u)) \frac{du}{u} dt.$$

The function α should be well behaved: $\alpha(y) = \alpha(1) + O(\log(y)) > \frac{1}{2}\alpha(1)$.

b) Show that

$$\int_{\mathbb{R}} t\varphi(t) \left(\frac{l_1}{l_2}\right)^{it} dt \ll \min(T^2, \log(l_1/l_2)^{-2})$$

and deduce that

$$S_1(T,L,N) \ll T(T+L)\|a_n\|_2^2 \|b_l\|_2^2.$$

c) Show that $\Phi(x,y) \ll T^{-1}$ for $|\log y| \ll 1$ and $x \ll T$. Use this to prove

$$S_2(T,L,N) \ll T^{1+\epsilon} L \|a_n\|_2^2 \|b_l\|_2^2.$$

This is the final missing piece to complete the exercise.

Exercise 2.3: An interesting application of the backward Kuznetsov formula and the results from Exercise 2.1 and 2.2 is the following fourth moment of zeta. Let $T \geq 2$, $T^{\frac{1}{2}} < T_0 \leq T$ and $T \leq t_1 < t_2 < \ldots < t_R \leq 2T$ with $t_{r+1} - t_r \geq T_0$. Then

$$\sum_{r=1}^{R} \int_{t_r}^{t_r+T_0} |\zeta(\frac{1}{2}+it)|^4 dt \ll (RT_0 + R^{\frac{1}{2}} T_0^{-\frac{1}{2}} T) T^{\epsilon}. \tag{6}$$

Choosing $R = 1$ and $T_0 = T^{\frac{2}{3}}$ yields

$$\int_T^{T+T^{\frac{2}{3}}} |\zeta(\frac{1}{2}+it)|^4 dt \ll T^{\frac{2}{3}+\epsilon}.$$

Another nice corollary is the bound

$$\int_0^T |\zeta(\frac{1}{2}+it)|^{12} dt \ll T^{2+\epsilon}$$

originally due to Heath-Brown. We will now sketch the proof of (6) up to the point where Kloosterman sums come into play. Then the exercises will start.

We set

$$M(\frac{1}{2}+it) = \sum_{m\in\mathbb{Z}} \alpha(m) m^{-\frac{1}{2}-it}$$

for a smooth function α with support in $[M, 2M]$, with $M < T^{\frac{1}{2}} \log(T)$. Further we require $\alpha^{(p)}(m) \ll_p M^{-p}$. By the typical approximate functional equation yoga we can reduce the problem to showing that

$$\sum_{r=1}^{R} \int_{t_r}^{t_r+T_0} |\zeta(\frac{1}{2}+it)|^2 |M(\frac{1}{2}+it)|^2 dt \ll (RT_0 + R^{\frac{1}{2}} T_0^{-\frac{1}{2}} T) T^{\epsilon}.$$

Let $\theta(x)$ be a positive smooth function with support in $[-2, 2]$ dominating the indicator function on the unit interval. Set $f(x) = \theta(Tx/T_0)$ and $j(\tau) = \theta(\tau/T - 1)$. Opening $|M(\frac{1}{2}+it)|^2$, sorting the resulting double sum by greatest common divisor and including the test functions f and j shows that it suffices to estimate

$$\sum_{\tau=t_1,\ldots,t_R} j(\tau) \int_0^\infty e^{-\frac{2\pi t}{T}} f(\frac{t-\tau}{T}) |\zeta(\frac{1}{2}+it)|^2 G(t) dt \ll (RT_0 + R^{\frac{1}{2}} T_0^{-\frac{1}{2}} T) T^{\epsilon},$$

where

$$G(t) = \sum_{(h,k)=1} (hk)^{-\frac{1}{2}} \left(\frac{h}{k}\right)^{it} g(h) g(k)$$

for any positive smooth function g with support in $[M, 4M]$ for $1 \leq M \leq T^{\frac{1}{2}} \log(T)$ satisfying $g^{(p)}(x) \ll_p M^{-p}$.

Using Fourier inversion we write

$$\int_0^\infty e^{-\frac{2\pi t}{T}} f(\frac{t-\tau}{T})|\zeta(\frac{1}{2}+it)|^2 G(t)dt = 2\Re \int_0^\infty \hat{f}(v)e(-\frac{v\tau}{T})W(v)dv \tag{7}$$

where

$$W(v) = \int_0^\infty e(\frac{v+i}{T}t)|\zeta(\frac{1}{2}+it)|^2 G(t)dt.$$

Partial integration gives the bound $\hat{f}(v) \ll_p \frac{T_0}{T}(\frac{T}{(|v|+1)T_0})^p$, which we should keep in mind. At this point classical results give the bound $W(v) \ll T^{1+\epsilon}$, which suffices to treat the ranges $v < T^\epsilon$ and $v > T^{1+\epsilon}/T_0$. By a common dyadic dissection one reduces the problem to showing

$$\sum_{\tau=t_1,\dots,t_R} j(\tau)S(M,N,\tau) \ll (RT_0 + R^{\frac{1}{2}}T_0^{-\frac{1}{2}}T)T^\epsilon$$

for the integrals

$$S(M,N,\tau) = \int_{N/2}^N \hat{f}(v)S(\log(v))e(-\frac{v\tau}{T})W(v)dv$$

with $1 \leq M \leq T^{\frac{1}{2}}\log(T)$ and $T^\epsilon < N \leq T^{1+\epsilon}T_0^{-1}$. We can assume that the integrand (and all its derivatives) vanishes at the end points of the integral.
A Lemma of Titchmarsh gives the Laplace transform of $|\zeta(\frac{1}{2}+it)|^2$ as

$$\int_0^\infty e^{-zt}|\zeta(\frac{1}{2}+it)|^2 dt = 2\pi e^{\frac{iz}{2}} \sum_{l=1}^\infty \tau(l)\exp(2\pi i l e^{iz}) + p(z)$$

for $\Re(z) > 0$ and a function p, which is regular for z sufficiently close to 0. Define

$$S(z,\frac{h}{k}) = \sum_{l=1}^\infty \tau(l)e(l\frac{h}{k})\exp(-lz).$$

Recalling the definition of $G(t)$ and inserting Titchmarsh's result yields

$$W(v) = 2\pi\exp(\pi\frac{v+i}{T}) \sum_{(h,k)=1} \frac{1}{k}g(h)g(k)S(2\pi i\frac{h}{k}(1-e^{2\pi(v+i)/T}),\frac{h}{k}) + O(M).$$

The sum $S(z,\frac{h}{k})$ can be written as the Mellin transform

$$S(z,\frac{h}{k}) = \frac{1}{2\pi i}\int_{(c)} D(s,\frac{h}{k})\Gamma(s)z^{-s}ds$$

of the (Estermann-type) zeta function

$$D(s;\frac{h}{k}) = \sum_{l=1}^\infty \tau(l)e(-l\frac{h}{k})l^{-s}, \text{ for } \Re(s) > 1.$$

The function $D(s, \frac{h}{k})$ extends meromorphically to $\mathbb{C}$ and has a pole of order 2 at $s = 1$. The Laurent expansion at $s = 1$ is

$$D(s, \frac{h}{k}) = \frac{1}{k}(s-1)^{-2} + \frac{2}{k}(\gamma - \log(k))(s-1)^{-1} + \dots.$$

Further we have the nice functional equation

$$D(s, \frac{h}{k}) = 2(2\pi)^{2s-2}\Gamma(1-s)^2 k^{1-2s}[D(1-s; \frac{\overline{h}}{k}) - \cos(\pi s) D(1-s; -\frac{\overline{h}}{k})].$$

Finally, one has the bound $|D(0, \frac{h}{k})| \leq k\log(2k)^2$. Using these properties (due to Estermann) together with typical contour shift arguments yields the decomposition

$$S(z; \frac{h}{k}) = R_0(T; \frac{h}{k}) + R_1(T, v; h, k) + R_2(T, v; \frac{h}{k}) + R_3(T, v; \frac{h}{k}) \tag{8}$$

with

$$\begin{aligned}
R_0(T; \frac{h}{k}) &= D(0; \frac{h}{k}),\\
R_1(T, v; h, k) &= \frac{1}{zk}(\gamma - 2\log(k) - \log(z)),\\
R_2(T, v; \frac{h}{k}) &= \frac{1}{2\pi i}\int_{(1-c)} 2(2\pi)^{2s-2}\Gamma(1-s)^2\Gamma(s)k^{1-2s}[D(1-s; \frac{\overline{h}}{k}) - e^{-\pi i s}D(1-s; -\frac{\overline{h}}{k})]z^{-s}ds \text{ and}\\
R_3(T, v; \frac{h}{k}) &= -\frac{1}{\pi}\int_{(1-c)} 2(2\pi)^{2s-2}\Gamma(1-s)^2\Gamma(s)\sin(\pi s)k^{1-2s}D(1-s; -\frac{\overline{h}}{k})z^{-s}ds.
\end{aligned}$$

The total contribution of $R_0(T; \frac{h}{k})$ to $S(M, NM\tau)$ is

$$\begin{aligned}
S_0(M, N, \tau) = 2\pi\int_{N/2}^{N} \hat{f}(v)S(\log(v))e(-\frac{v\tau}{T})\exp(\pi\frac{v+i}{T})dv \sum_{(h,k)=1}\frac{1}{k}g(h)g(k)D(0; \frac{h}{k})\\
\ll T_0T^{\epsilon},
\end{aligned}$$

simply by integration by parts. Similarly easy one can estimate the contribution of $R_1(T, v; \frac{h}{k})$ to $S(M, N, \tau)$ by

$$\begin{aligned}
S_0(M, N, \tau) = 2\pi\int_{N/2}^{N} \hat{f}(v)S(\log(v))e(-\frac{v\tau}{T})\exp(\pi\frac{v+i}{T})dv \sum_{(h,k)=1}\frac{1}{k}g(h)g(k)R_1(T, v; \frac{h}{k})\\
\ll T_0\log(T),
\end{aligned}$$

The contribution of $R_2(T, v; \frac{h}{k})$ is handled using Stirling's formula and trivial bounds for $|D(1-s; \frac{h}{k})| \leq \zeta^2(c)$ on the contour $(1-c)$. One can deduce

$$S_2(M, N; \tau) \ll T_0.$$

It remains to bring $R_3(T,v;\frac{h}{k})$ in shape. We expand $D(1-s;-\frac{\overline{h}}{k})$ into its Dirichlet series, recall the duplication formula $\Gamma(1-s)\Gamma(s)\sin(\pi s)=\pi$ as well as the Mellin integral $\frac{1}{2\pi i}\int_{(\sigma)}\Gamma(w)x^{-w}dx=\exp(-x)$. With this we can rewrite

$$R_3(T,v;\frac{h}{k})=-\frac{2\pi i}{zk}\sum_{l=1}^{\infty}\tau(l)e(-l\frac{\overline{h}}{k})\exp(-\frac{4\pi^2 l}{zk^2}),\text{ with } z=2\pi i\frac{h}{k}(1-e^{2\pi(v+i)/T}.$$

Gathering everything gives the contribution

$$S_3(M,N,\tau)=\pi\int_{N/2}^{N}\hat{f}(v)S(\log(v))\sinh(\pi\frac{v+i}{T})^{-1}e(-\frac{v\tau}{T}) \cdot\sum_{l=1}\tau(l)\sum_{(h,k)=1}\frac{g(h)g(k)}{hk}e\left(-l\frac{\overline{h}}{k}-\frac{l}{hk}(e^{2\pi(v+i)/T}-1)^{-1}\right)dv$$

of $R_3(T,v;\frac{h}{k})$ to $S(M,N,\tau)$.
Put $L=2\pi M^2N^2T^{-1}$. The part of $S_3(M,N,\tau)$ where l lies outside the interval $[L/16,16L]$ can be estimated trivially using partial integration (for the v-integral). This can be detected using a positive smooth function $b(x)$ supported in $[L/32,32L]$ that satisfies $b^{(b)}(x)\ll x^{-p}$ and dominates the indicator function on $[L/16,16L]$. We get

$$S(M,,\tau)=S_4(M,N,\tau)+O(T_0T^{\epsilon})$$

with

$$S_4(M,N,\tau)=\sum_{l}\tau(l)\sum_{(h,k)=1}C(h,k,l,\tau)e(-l\frac{\overline{h}}{k}) \qquad (9)$$

and

$$C(h,k,l,\tau)=\pi b(l)\frac{g(h)g(k)}{hk}\int_{N/2}^{N}\hat{f}(v)S(\log(v))\sinh(\pi(v+i)/T)^{-1}e\left(-v\frac{\tau}{T}-\frac{l}{hk}(e^{2\pi(v+i)/T}-1)^{-1}\right)dv.$$

Using the method of stationary phase (for the v-integral) one can show that

$$j(\tau)S_4(M,N,\tau)=M^{-1}N^{-\frac{1}{2}}T_0\sum_{l}\tau(l)\sum_{(h,k)=1}\frac{1}{k}b(h,k,l,\tau)e\left(-l\frac{\overline{h}}{k}-\left(\frac{2lT^2}{\pi hk\tau}\right)^{\frac{1}{2}}\right).$$

for a smooth function $b(x_1,x_2,x_3,x_4)$, which is non-zero only for

$$x_1,x_2\asymp M,\ x_3\asymp L\text{ and }x_4\asymp T$$

and satisfies

$$\frac{\partial^{|\mathbf{j}|}}{\partial\mathbf{x}^{\mathbf{j}}}b(\mathbf{x})\ll_{\mathbf{j}}\mathbf{x}^{-\mathbf{j}},$$

where we use standard multi index notation.
Put

$$H(x,k,l,\tau)=b(x,k,l,\tau)e\left(-\left(\frac{2lT^2}{\pi hk\tau}\right)^{\frac{1}{2}}\right).$$

We can expand this in a Fourier series

$$H(x,k,l,\tau) = \sum_{u\in\mathbb{Z}} \hat{H}(u,k,l,\tau)e(u\frac{x}{k})$$

with

$$\hat{H}(u,k,l,\tau) = \frac{1}{k}\int_{\mathbb{R}} b(x,k,l,\tau)e(-u\frac{x}{k} - \left(\frac{2lT^2}{\pi hk\tau}\right)^{\frac{1}{2}})dx.$$

Another application of the method of stationary phase leads to

$$j(\tau)S_4(M,N,\tau) = \frac{T_0}{MN}\sum_{u,l,k}\tau(l)\frac{1}{k}a(k,l,u,\tau)S(u,-l;k)e(-3\left(\frac{ulT^2}{2\pi k^2\tau}\right)^{\frac{1}{3}}) + O(T_0). \qquad (10)$$

Now $a(x_1,x_2,x_3,x_4)$ is a smooth function non-vanishing only for

$$x_1 \asymp M,\ x_2 \asymp L,\ x_3 \asymp N \text{ and } x_4 \asymp T$$

satisfying

$$\frac{\partial^{|\mathbf{j}|}}{\partial \mathbf{x}^{\mathbf{j}}}a(\mathbf{x}) \ll_{\mathbf{j}} \mathbf{x}^{-\mathbf{j}}.$$

Kloosterman sums finally arrived on stage for the grand finale. Note that estimating trivially using the Weil bound suffices for $T_0 \geq T^{\frac{7}{8}}$.

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a) Put $f(x) = a(\frac{4\pi\sqrt{ul}}{x}, l, u, \tau)e(-\frac{3}{2\pi}\left(\frac{T^2x^2}{4\tau}\right)^{\frac{1}{3}})$. Recall the Bessel transform $\tilde{f}$ from (1) and show that

$$\tilde{f}(r) = \theta(r)c(l,u,\tau,r)\tau^{ir} + O((|r|+N)^{-6}e^{-\pi|r|}\log(T)),$$

where $|\theta(r)| \leq r^{-1}e^{-\pi r}$. Furthermore $c(x_1,x_2,x_3,x_4)$ is a smooth function non-vanishing only for

$$x_1 \asymp L,\ x_2 \asymp N,\ x_3 \asymp T \text{ and } x_4 \asymp N$$

satisfying $\frac{\partial^{|\mathbf{j}|}}{\partial \mathbf{x}^{\mathbf{j}}}c(\mathbf{x}) \ll_{\mathbf{j}} \mathbf{x}^{-\mathbf{j}}$. (Recall that $L = 2\pi M^2N^2T^{-1}$, $M < T^{\frac{1}{2}}\log(T)$ and $T^{\epsilon} < N < T^{1+\epsilon}T_0^{-1}$.)

b) Locate the function f from a) in our expression for $S_4(M,N,\tau)$ and apply the backward Kuznetsov formula to the k-sum. Derive

$$\begin{aligned}\sum_{v=1}^{R} j(t_v)S_4(M,N,t_v) \ll & RT_0T^{\epsilon} \\ & + \frac{T_0}{MN}\int\left|\sum_{v=1}^{R}\sum_{u,l}\tau(l)(ul)^{ir}\sigma_{2ir}(u)\sigma_{2ir}(l)t_v^{ir}c(l,u,t_v,r)\right|\frac{dr}{r|\zeta(1+2ir)|^2} \\ & + \frac{T_0}{MN}\sum_{j=1}^{\infty}\frac{1}{t_j\cosh(\pi t_j)}\left|\sum_{v=1}^{R}\sum_{u,l}\tau(l)\rho_j(u)\rho_j(l)t_v^{it_j}c(l,u,t_v,t_j)\right|.\end{aligned}$$

c) Apply the results from Exercise 2.1 and 2.2 to derive the desired bound.

List of useful facts:

- The classical Rankin-Selberg estimate
$$\sum_{n\leq N}|\rho_j(n)|^2 \ll (|t_j|+1)^{\epsilon}\cosh(\pi t_j)N.$$
- The Weyl law
$$\sharp\{j\in\mathbb{N}\colon t_j\leq X\}=\frac{1}{12}X^2+O(X)$$
as well as the estimate
$$\sharp\{j\in\mathbb{N}\colon |t-t_j|\leq 1\}\ll t.$$
- One also has the (elementary) identity
$$S(n,m;c)=\sum_{d|(n,m,c)} d\cdot S\left(1,\frac{nm}{d^2},\frac{c}{d}\right).$$
The Weil bound for Kloosterman sums is
$$|S(n,m;c)|\leq (n,m,c)^{\frac{1}{2}}c^{\frac{1}{2}}\tau(c).$$
- The large sieve inequality
$$\sum_{h \bmod c}\left|\sum_{m\leq M}a_m e(m\frac{h}{c})\right|^2\leq (c+M)\sum_{m\leq M}|a_m|^2.$$
- For $x>0$ and $-1<\Re(v)<1$ one has
$$\pi J_v(x)=2\int_0^{\infty}\sin(x\cosh t-\frac{\pi v}{2})\cosh(vt)dt.$$
- Basset's formula reads
$$K_{2ir}(x)=\frac{3}{\sqrt{\pi}}\Gamma(\frac{1}{2}+2ir)\int_0^{\infty}\left(\frac{2x}{w^2+x^2}\right)^{2ir}\frac{w\sin(w)}{(x^2+w^2)^{\frac{3}{2}}}dw.$$
Furthermore $K_{2ir}(x)=\frac{\pi}{2}\frac{1}{\sinh(2\pi r)}(I_{2ir}(x)-I_{-2ir}(x))$ and we have the power series expansion
$$I_{2ir}=(x/2)^{2ir}\frac{1}{\Gamma(1+2ir)}\sum_{m=0}^{\infty}\frac{(x/2)^{2m}}{m!}(1+2ir)\cdots(m+2ir).$$
- A mean-value theorem for Dirichlet polynomials:
$$\int_{r\asymp N}\Big|\sum_{v=1}^{R}t_v^{ir}e(xt_v)\Big|^2dr\ll (N+\frac{T}{T_0})R,$$
where the t_v are of size T and are at least T_0 apart.

A version of the stationary phase lemma:

Let $\mathbf{D}$ be the domain in $\mathbb{R}^{r+1}$ given by $(1-c_i)X_i < x_i < (1+c_i)X_i$ for $1 \leq i \leq r$ and $(1-c)Y < y < (1+c)Y$ for some constants $0 < c_1, \ldots, c_r, c < \frac{1}{2}$. Further, let $f(x_1, \ldots, x_r, y)$ be a real function in $\mathcal{C}^\infty(\mathbf{D})$ satisfying the following properties:

- For $(\mathbf{x}, y) \in \mathbf{D}$ we have $\Delta \leq \frac{\partial^2}{\partial y^2} f(\mathbf{x}, y) \leq 2\Delta$;
- For $(\mathbf{x}, y) \in \mathbf{D}$ we have

$$\Big| \frac{\partial^{|p|+q}}{\partial \mathbf{x}^p \partial y^q} f(\mathbf{x}, y) \Big| \leq c(p,q) \Delta Y^{2-q} X_1^{-p_1} \cdots X_r^{-p_r},$$

  for some constants $c(p,q)$ so that $c(0,3) \leq \frac{1}{4c}$.
- The equation $\frac{\partial}{\partial y} f(\mathbf{x}, y) = 0$ has a smooth solution $y = y(\mathbf{x})$ in $\mathbf{D}$ so that

$$\frac{\partial |p|}{\partial \mathbf{x}^p} \ll_p X_1^{-p_1} \cdots X_r^{-p_r}.$$

Finally let $a(\mathbf{x}, y)$ be a function smooth function with $\operatorname{supp}(a) \subseteq \mathbf{D}$ satisfying $\frac{\partial^{|p|+q}}{\partial \mathbf{x}^p \partial y^q} a(\mathbf{x}, y) \ll_{p,q} X_1^{-p_1} \cdots X_r^{-p_r} Y^{-q}$. Then

$$\int_{\mathbb{R}} b(\mathbf{x}, y) e(f(\mathbf{x}, y)) dy = \Delta^{-\frac{1}{2}} b(\mathbf{x}) e(f(\mathbf{x}, y(\mathbf{x}))),$$

where b is a smooth function supported in the ranges

$$x_i \asymp X_i \text{ for } i = 1, \ldots, r$$

satisfying

$$\frac{\partial^p}{\partial \mathbf{x}^p} b(\mathbf{x}) \ll X_1^{-p_1} \cdots X_r^{-p_r}.$$

Remark: *This is a version of the stationary phase lemma featuring non-degenerate critical points and a set of r parameters. There are many versions of this result in the literature. Important for our application is that we do not lose control over the new weight function $b(\mathbf{x})$.*