POD for Parametric PDEs and for Optimality Systems

M. Kahlbacher, K. Kunisch, H. Müller and S. Volkwein

Institute for Mathematics and Scientific Computing
University of Graz, Austria

DMV-Jahrestagung 2006, Bonn

Motivation 1: Parameter identification

- Model equations:

$$(*)\qquad \begin{aligned} -\mathrm{div}\big(c\nabla u\big)+\beta\cdot\nabla u+au &= f && \text{in } \Omega\subset\mathbb{R}^d\\ c\,\frac{\partial u}{\partial n}+qu &= g_N && \text{on } \Gamma_N\subset\Gamma\\ u &= g_D && \text{on } \Gamma_D=\Gamma\setminus\Gamma_N \end{aligned}$$

- Problem: estimate parameters (e.g., β or a) in $(*)$ from given (perturbed) measurements u_d for the solution u on (parts of) Γ
- Mathematical formulation: (∞-dim.) optimization problem

$$\min \int_\Gamma \alpha\,|u-u_d|^2\,\mathrm{d}s+\kappa\,\|p\|^2 \quad \text{s.t.} \quad (p,u) \text{ solves } (*) \text{ and } p\in P_{\mathrm{ad}}$$

- Numerical strategy: combine optimization methods with fast (local) rate of convergence and POD model reduction for the PDEs

s.t. — subject to

Motivation 2: Optimal control of time-dependent problems

- Model problem:

$$\min \frac{1}{2}\int_\Omega |y(T)-y_T|^2\,\mathrm{d}x + \frac{\kappa}{2}\int_0^T\int_\Gamma |u|^2\,\mathrm{d}x\mathrm{d}t$$

$$\text{s.t.}\quad \begin{cases} y_t - \Delta y + f(y) &= 0 \\ y|_\Gamma &= u \\ y(0) &= y_\circ \end{cases}$$

- Adjoint system:

$$-p_t - \Delta p + f'(y)^* p = 0, \quad p|_\Gamma = 0, \quad p(T) = y_T - y(T)$$

- Optimizer: second-order algorithms like SQP or Newton methods
- Challenge: large-scale $\leftrightarrow$ fast/real-time optimizer

Motivation 3: Closed-loop control for time-dependent PDEs

- Open-loop control:

$$\text{input } u(t) \longrightarrow \boxed{\begin{array}{l} \dot{y}(t)=f(t,y(t),u(t)) \\ y(0)=y_\circ\in\mathbb{R}^\ell \\ \text{(after spatial discretization)} \end{array}} \longrightarrow \text{output } y(t)$$

- Closed-loop control: determine $\mathcal{F}$ with

$$u(t) = \mathcal{F}(t, y(t)) \quad \text{(feedback law)}$$

- Linear case: LQR and LQG design
- Nonlinear case: Hamilton-Jacobi-Bellman equation

$$v_t(t, y_\circ) + H(v_y(t, y_\circ), y_\circ) = 0 \quad \text{in } (0, T) \times \mathbb{R}^\ell$$

- Strategy: ℓ-dim. spatial approximation by POD model reduction

Outline of the talk

- POD in Hilbert spaces
- Parameter estimation in elliptic systems
- POD for optimality systems (OS-POD)
- Conclusions

POD in Hilbert spaces

- Topology: Hilbert space X with inner product $\langle \cdot\,, \cdot \rangle$
- Snapshots: $y_1, \dots, y_n \in X$
- Snapshot ensemble: $\mathcal{V} = \mathrm{span}\,\{y_1, \dots, y_n\} \subset X$, $d = \dim \mathcal{V} \le n$
- POD basis of any rank $\ell \in \{1, \dots, d\}$: with weights $\alpha_j \ge 0$

$$\min \sum_{j=1}^{n} \alpha_j \Big\| y_j - \sum_{i=1}^{\ell} \langle y_j, \psi_i \rangle \psi_i \Big\|^2 \quad \text{s.t.} \quad \langle \psi_i, \psi_j \rangle = \delta_{ij}$$

- Constrained optimization:

$$\min J(\psi_1, \dots, \psi_\ell) \quad \text{s.t.} \quad \langle \psi_i, \psi_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

s.t. — subject to

Optimality conditions and computation of POD basis

- EVP for linear, symmetric $\mathcal{R}^n$ in X:

$$\mathcal{R}^n u_i = \sum_{j=1}^{n} \alpha_j \langle u_i, y_j \rangle y_j = \lambda_i u_i$$

and set $\psi_i = u_i$

- EVP for linear, symmetric $\mathcal{K}^n = ((\langle y_j, y_i \rangle))$ in $\mathbb{R}^n$:

$$\mathcal{K}^n v_i = \lambda_i v_i$$

and set $\psi_i = \frac{1}{\sqrt{\lambda_i}} \sum_{j=1}^{n} \alpha_j (v_i)_j y_j$ (methods of snapshots)

- Error for the POD basis of rank ℓ:

$$\sum_{j=1}^{n} \alpha_j \Big\| y_j - \sum_{i=1}^{\ell} \langle y_j, \psi_i \rangle \psi_i \Big\|^2 = \sum_{i=\ell+1}^{d} \lambda_i$$

Properties of the POD basis

- **Uncorrelated POD coefficients**:

$$\sum_{j=1}^{n} \alpha_j \langle y_j, \psi_i \rangle \langle y_j, \psi_k \rangle = \delta_{ik} \lambda_i$$

- **Optimality of the POD basis**:

$$\sum_{i=1}^{\ell} \sum_{j=1}^{n} \alpha_j \big| \langle y_j, \psi_i \rangle \big|^2 \geq \sum_{i=1}^{\ell} \sum_{j=1}^{n} \alpha_j \big| \langle y_j, \chi_i \rangle \big|^2$$

where $\{\chi_i\}_{i=1}^{\ell}$ orthonormal in X

POD for λ-ω systems [Müller/V.]

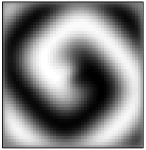
- PDEs: $s = u^2 + v^2$, $\lambda(s) = 1 - s$, $\omega(s) = -\beta s$

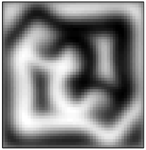
$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \lambda(s) & -\omega(s) \\ \omega(s) & \lambda(s) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \sigma\Delta u \\ \sigma\Delta v \end{pmatrix}$$

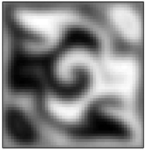
- Homogeneous boundary conditions:

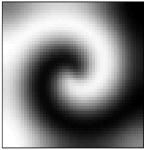
$$u = v = 0 \quad \text{or} \quad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0$$

- Initial conditions: $u_\circ(x_1, x_2) = x_2 - 0.5$, $v_\circ(x_1, x_2) = (x_1 - 0.5)/2$

u for β=1 and t=100


u for β=2 and t=100


u for β=3 and t=100


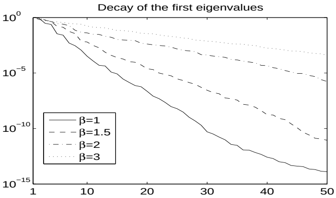
u for β=1 and t=100


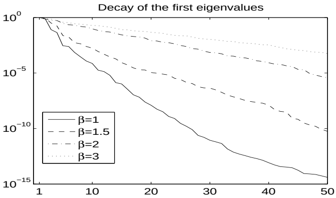
POD basis for λ-ω systems

- Offsets: $\bar{u}(x) = \frac{1}{n}\sum_{j=1}^{n} u(t_j, x)$ or $\bar{u} \equiv 0$
- Snapshots: $\hat{u}_j(x) = u(t_j, x) - \bar{u}(x)$ for $1 \le j \le n$
- POD eigenvalue problem: $X = L^2(\Omega)$

$$\mathcal{K} v_i = \lambda v_i, \; 1 \le i \le \ell, \quad \text{with } \mathcal{K}_{ij} = \int_\Omega \hat{u}_j(x)\hat{u}_i(x)\,\mathrm{d}x$$

- POD basis computation: $\psi_i = \frac{1}{\sqrt{\lambda_i}} \sum_{j=1}^{n} \alpha_j (v_i)_j \hat{u}_j$





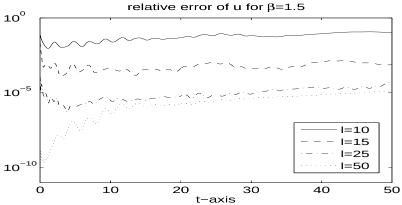
ROM for λ-ω systems

- POD Galerkin ansatz:

$$u_\ell(t,x) = \bar u(x) + \sum_{j=1}^{\ell} u_\ell^j(t)\psi_j(x), \quad v_\ell(t,x) = \bar v(x) + \sum_{j=1}^{\ell} v_\ell^j(t)\phi_j(x)$$

- Reduced-order model (ROM):
  - insert ansatz into PDEs
  - multiply by POD basis functions ψ_i respectively ϕ_i
  - integrate over Ω
- Numerical results:

$$t \mapsto \frac{\|u_\ell(t) - u(t)\|^2_{L^2(\Omega)}}{\|u(t)\|^2_{L^2(\Omega)}}$$



Relative POD errors for λ-ω systems

- Offsets: $u_{\mathrm{m}}(x) = \frac{1}{n}\sum_{j=1}^{n} u(t_j, x)$ or $\bar{u} \equiv 0$
- Relative POD errors:

	$\bar{u} = 0$	$\bar{u} = u_{\mathrm{m}}$
$\ell = 10$	0.005890	0.005945
$\ell = 15$	0.000350	0.000335
$\ell = 50$	0.000009	0.000009

	$\bar{u} = 0$	$\bar{u} = u_{\mathrm{m}}$
$\ell = 40$	0.577442	0.460188
$\ell = 45$	0.898613	0.297619
$\ell = 50$	0.071035	0.001774

$$E_{\mathrm{rel}}(u) = \frac{\sum_{j=1}^{n} \alpha_j \, \|u_\ell(t_j) - u(t_j)\|^2_{L^2(\Omega)}}{\sum_{j=1}^{n} \alpha_j \, \|u(t_j)\|^2_{L^2(\Omega)}} \quad \text{for } \beta = 1.5 \text{ (left) and } \beta = 2 \text{ (right)}$$

Continuous POD in Hilbert spaces [Henri/Yvon, Kunisch/V., ...]

- Snapshots: $y(\mu) \in X$ for all $\mu \in \mathcal{I}$
- Snapshot ensemble: $\mathcal{V} = \{y(\mu) \,|\, \mu \in \mathcal{I}\} \subset X$, $d = \dim \mathcal{V} \leq \infty$
- POD basis of rank $\ell < d$:

$$\min \int_{\mathcal{I}} \Big\| y(\mu) - \sum_{i=1}^{\ell} \langle y(\mu), \psi_i \rangle \, \psi_i \Big\|^2 \mathrm{d}\mu \quad \text{s.t.} \quad \langle \psi_i, \psi_j \rangle = \delta_{ij}$$

- EVP for linear, symmetric $\mathcal{R}$ in X:

$$\mathcal{R}\psi_i^\infty = \int_{\mathcal{I}} \langle \psi_i^\infty, y(\mu) \rangle \, y(\mu) \, \mathrm{d}\mu = \lambda_i^\infty \psi_i^\infty$$

- Error for the POD basis of rank ℓ:

$$\int_{\mathcal{I}} \Big\| y(\mu) - \sum_{i=1}^{\ell} \langle y(\mu), \psi_i^\infty \rangle, \psi_i^\infty \Big\|^2 \mathrm{d}\mu = \sum_{i=\ell+1}^{\infty} \lambda_i^\infty$$

Relationship between 'discrete' and continuous POD

- Operators $\mathcal{R}^n$ and $\mathcal{R}$:

$$\mathcal{R}^n\psi = \sum_{j=1}^{n} \alpha_j \, \langle \psi, y(\mu_j)\rangle y(\mu_j) \qquad \text{for } \psi \in X = L^2(\Omega)$$

$$\mathcal{R}\psi = \int_{\mathcal{I}} \langle \psi, y(\mu)\rangle y(\mu) \, \mathrm{d}\mu \qquad \text{for } \psi \in X = L^2(\Omega)$$

- Operator convergence of $\mathcal{R}^n - \mathcal{R}$: y smooth and appropriate α_j's
- Perturbation theory [Kato]: $(\lambda_i, \psi_i) \overset{n\to\infty}{\longrightarrow} (\lambda_i^\infty, \psi_i^\infty)$ for $1 \le i \le \ell$
- Choice of the weights α_j?: ensure convergence $\mathcal{R}^n \overset{n\to\infty}{\longrightarrow} \mathcal{R}$

Parameter estimation [Kahlbacher/V.]

- Model equations: $\beta(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$-2\Delta u + \beta \cdot \nabla u + a u = 1 \qquad \text{in } \Omega = (0,1)\times(0,1)$$
$$2\,\frac{\partial u}{\partial n} + \frac{3}{2}\,u = -1 \qquad \text{on } \Gamma$$

- Snapshots: (FE) solutions $\{u_j\}_{j=1}^{102}$ for $a_j = -51.5 + j$
- POD basis of rank ℓ:

$$(\mathbf{P}^\ell) \qquad \min \sum_{j=1}^{102} \Big\| u_j - \sum_{i=1}^{\ell} \langle u_j, \psi_i\rangle\, \psi_i \Big\| \quad \text{s.t.} \quad \langle \psi_i, \psi_j\rangle = \delta_{ij}$$

with $\langle \varphi, \phi\rangle = \int_\Omega \varphi\phi\,\mathrm{d}x$ and $\|\varphi\| = \sqrt{\langle \varphi, \varphi\rangle}$

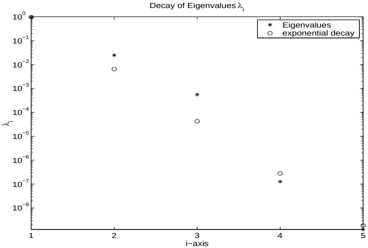
- Solution to $(\mathbf{P}^\ell)$: correlation matrix $K_{ij} = \langle u_i, u_j\rangle$

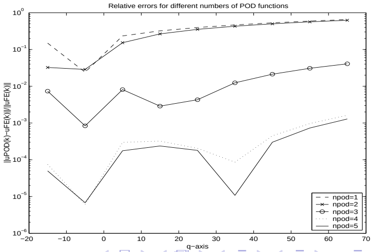
$$K v_i = \lambda_i v_i, \quad \lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_\ell, \quad \psi_i = \frac{1}{\sqrt{\lambda_i}} \sum_{j=1}^{102} (v_i)_j u_j$$

Reduced-order modelling (ROM)

- **Ansatz:** $u^\ell = \sum_{i\le\ell} u_i^\ell \psi_i$ and Galerkin projection
- **Error estimate:** $\int \|u^\ell(a) - u(a)\|^2 \,\mathrm{d}a \sim \sum_{i>\ell} \lambda_i$
- **Exponential decay of the eigenvalues:** $\lambda_i = \lambda_1 e^{-\eta(i-1)}$
- **Experimental order of decay (EOD)** [Hinze/V.]:

$$EOD := \frac{1}{\ell_{\max}} \sum_{\ell=1}^{\ell_{\max}} Q(\ell) \quad \text{with} \quad Q(\ell) = \ln \frac{\int \|u^\ell(a)-u(a)\|^2 \,\mathrm{d}a}{\int \|u^{\ell+1}(a)-u(a)\|^2 \,\mathrm{d}a} \sim \eta$$





Parameter identification

- Model equations: $g(x) = x_1$

$$(*)\qquad \begin{aligned} -\tfrac{3}{4}\,\Delta u + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \nabla u + au = g &\quad \text{in } \Omega = (0,1)\times(0,1) \\ \tfrac{3}{4}\,\tfrac{\partial u}{\partial n} + \tfrac{3}{2}\,u = -1 &\quad \text{on } \Gamma \end{aligned}$$

- Data: choose $a_{\rm id} \geq 0$ and compute (FE) solution $u(a_{\rm id})$ to $(*)$
- Reconstruction: estimate $a \geq 0$ from $u_d = (1+\varepsilon\delta)u(a_{\rm id})\big|_\Gamma$ with random $|\varepsilon| \leq 1$ and factor $\delta = 5\%$
- Constrained optimization:

$$\min J(a,u) = \int_\Gamma \alpha\,|u-u_d|^2\,\mathrm{d}s + \kappa\,|a|^2 \text{ s.t. } (a,u) \text{ solves } (*) \text{ and } a \geq 0$$

- Relaxation of the inequality:

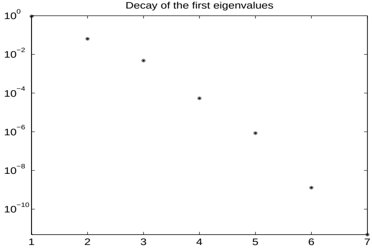
$$\min J_\lambda^\varrho(a,u) = J(a,u) + \tfrac{1}{\varrho}\max\big\{0, \lambda + \varrho(0-a)\big\}^2 \text{ s.t. } (a,u) \text{ solves } (*)$$

Global convergent optimization method

- Outer loop: augmented Lagrangian method → control of ϱ^k and λ^k
- Inner loop: globalized SQP algorithm with fixed (ϱ^k, λ^k) for

$$\min J_{\lambda^k}^{\varrho^k}(a,u) \quad \text{s.t.} \quad \begin{cases} -\frac{3}{4}\Delta u + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \nabla u + au = g & \text{in } \Omega \\ \frac{3}{4}\frac{\partial u}{\partial n} + \frac{3}{2} u = -1 & \text{on } \Gamma \end{cases}$$

- Numerical results: $\alpha = 5000$, $\kappa = 0.0005$, $a_{\rm id} = 25$, $\ell = 7$



relative errors:

$$\frac{\|u^\ell - u(a_{\rm id})\|}{\|u(a_{\rm id})\|} \approx 1.87 \cdot 10^{-5}$$

$$\frac{|a^\ell - a_{\rm id}|}{|a_{\rm id}|} \approx 6 \cdot 10^{-3}$$

OS-POD [Kunisch/V.]

- Minimize:

$$J(y,u) = \frac{\beta}{2}\int_0^T \|y(t)-z(t)\|_H^2\,\mathrm{d}t + \frac{1}{2}\int_0^T u(t)^T\mathbf{R}u(t)\,\mathrm{d}t$$

s.t. (e.g., Navier-Stokes)

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) + \mathcal{A}y(t) + \mathcal{N}(y(t)) = \sum_{k=1}^m u_k(t)b_k \text{ in } [0,T] \quad \text{and} \quad y(0) = y_\circ$$

- H, V Hilbert spaces, $V \hookrightarrow H = H' \hookrightarrow V'$ (e.g., $H = L^2$, $V = H^1$)
- $\mathbf{R} \in \mathbb{R}^{m\times m}$ with $\mathbf{R} \succ 0$, $z \in L^2(0,T;H)$, $\beta > 0$
- $a : V \times V \to \mathbb{R}$ bounded, symmetric, coercive
  $\mathcal{A} : V \to V'$ with $\langle \mathcal{A}\phi, \varphi\rangle_{V',V} = a(\phi,\varphi)$ for all $\phi, \varphi \in V$
- $\mathcal{N} : V \to V'$, $u \in L^2(0,T;\mathbb{R}^m)$, $y_\circ \in H$, $b_k \in H$

POD modelling

- Choose snapshots, e.g., $\mathcal{V} = \{y(t) \,|\, t \in [0, T]\}$ for fixed u
- Compute POD basis $\psi_1, \ldots, \psi_\ell$
- Galerkin ansatz: $x(t) = \sum_{i=1}^{\ell} x_i(t)\psi_i$
- Model reduction $\Rightarrow$ low dimension
- Problems:
  - Quality of the basis for unknown optimal control?
  - Can we avoid the computation of y for various inputs?
  - Can we avoid the computation of p for various observations?
- Compare: Balanced truncation for

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; t > 0 \quad \text{and} \quad x(0) = x_0$$
$$y(t) = Cx(t), \;\; t > 0$$

Optimality System-POD – 1

Minimize for fixed $\ell > 0$

$$J^\ell(x,\psi,u) = \frac{\beta}{2}\int_0^T x(t)^T\big(\mathbf{E}(\psi)x(t)-2z^\ell(t,\psi)\big)\,\mathrm{d}t+\frac{1}{2}\int_0^T u(t)^T\mathbf{R}u(t)\,\mathrm{d}t$$

s.t.

- $\mathbf{E}(\psi)\dot{x}(t) + \mathbf{A}(\psi)x(t) + \mathbf{N}(x(t),\psi) = \mathbf{B}(\psi)u(t)\ \forall t,\ \ \mathbf{E}(\psi)x(0) = x_\circ$

$\mathbf{E}_{ij} = \langle\psi_j,\psi_i\rangle_H$, $\mathbf{A}_{ij} = a(\psi_j,\psi_i)$, $\mathbf{B}_{ij} = \langle b_j,\psi_i\rangle_H$, $\mathbf{N}_i = \Big\langle\mathcal{N}\Big(\sum_{j=1}^{\ell} x_j\psi_j\Big),\psi_i\Big\rangle$, $z_i^\ell = \langle z(t),\psi_i\rangle_H$

Optimality System-POD – 1

Minimize for fixed $\ell > 0$

$$J^\ell(x,\psi,u) = \frac{\beta}{2}\int_0^T x(t)^T\big(\mathbf{E}(\psi)x(t) - 2z^\ell(t,\psi)\big)\,\mathrm{d}t + \frac{1}{2}\int_0^T u(t)^T\mathbf{R}u(t)\,\mathrm{d}t$$

s.t.

- $\mathbf{E}(\psi)\dot{x}(t) + \mathbf{A}(\psi)x(t) + \mathbf{N}(x(t),\psi) = \mathbf{B}(\psi)u(t)\ \forall t,\quad \mathbf{E}(\psi)x(0) = x_\circ$
- $\dot{y}(t) + \mathcal{A}y(t) + \mathcal{N}(y(t)) = (\mathcal{B}u)(t) = \sum_{k=1}^m u_k(t)b_k\ \forall t,\quad y(0) = y_\circ$
- $\mathcal{R}(y)\psi_i = \int_0^T \langle y(t),\psi_i\rangle y(t)\,\mathrm{d}t = \lambda_i\psi_i,\ \langle\psi_i,\psi_j\rangle = \delta_{ij}$

$\mathbf{E}_{ij} = \langle\psi_j,\psi_i\rangle_H$, $\mathbf{A}_{ij} = a(\psi_j,\psi_i)$, $\mathbf{B}_{ij} = \langle b_j,\psi_i\rangle_H$, $\mathbf{N}_i = \left\langle \mathcal{N}\Big(\sum_{j=1}^\ell x_j\psi_j\Big),\psi_i\right\rangle$, $z_i^\ell = \langle z(t),\psi_i\rangle_H$

OS-POD – 2

- Constraints:

$$\mathbf{E}(\psi)\dot{x}(t) + \mathbf{A}(\psi)x(t) + \mathbf{N}(x(t), \psi) = \mathbf{B}(\psi)u(t)\ \forall t,\ \ \mathbf{E}(\psi)x(0) = x_\circ$$

$$\dot{y}(t) + \mathcal{A}y(t) + \mathcal{N}(y(t)) = \sum_{k=1}^{m} u_k(t)b_k\ \forall t,\ \ y(0) = y_\circ$$

$$\mathcal{R}(y)\psi_i = \lambda_i\psi_i,\ \ \langle\psi_i, \psi_j\rangle = \delta_{ij}$$

- State variables: $w = (x, \psi_1, \ldots, \psi_\ell, y, \lambda_1, \ldots, \lambda_\ell)$
- POD basis: $\psi_i = \psi_i(u)$ computed from trajectory $y = y(u)$
- Existence of optimal solutions under assumptions for $\mathcal{N}$ and $\mathcal{A}$
- Existence of Lagrange multipliers provided $\lambda_1 > \ldots > \lambda_\ell > 0$

Optimality system for OS-POD

- Optimal solution: $w = (x, \psi_1, \dots, \psi_\ell, y, \lambda_1, \dots, \lambda_\ell)$
- Dual equation for x (POD) dynamics: $q : [0, T] \to \mathbb{R}^\ell$, $q(T) = 0$

$$-\mathbf{E}(\psi)\dot{q}(t) + \big(\mathbf{A}(\psi) + \mathbf{N}_x(x(t), \psi)^T\big) q(t) = \beta\big(z^\ell(t, \psi) - \mathbf{E}(\psi)x(t)\big)$$

- Dual equations for y dynamics: $p : [0, T] \to V$, $p(T) = 0$

$$-\dot{p}(t) + \left(\mathcal{A} + \mathcal{N}'(y(t))^\star\right) p(t) = \sum_{i=1}^{\ell} \Big(\langle y(t), \mu_i \rangle \psi_i + \langle y(t), \psi_i \rangle \mu_i \Big)$$

- Optimality condition: $\mathbf{R}u(t) = \mathbf{B}(\psi)^T q(t) + \mathcal{B}^\star p(t)$
- Right-hand side: $(\mathcal{R} - \lambda_i \mathcal{I})\mu_i = \mathcal{G}_i(w) \in \ker\, (\mathcal{R} - \lambda_i \mathcal{I})^\perp$, $1 \le i \le \ell$

Boundary control of the Burgers' equation

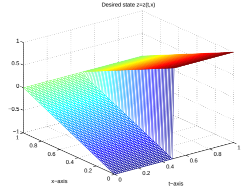
Consider

$$\min J(y,u) = \frac{1}{2}\int_0^T \left(\int_\Omega |y-z|^2 \,\mathrm{d}x + \beta \;\left(|u(t)|^2 + |v(t)|^2\right) \right) \mathrm{d}t$$

s.t.

$$\begin{aligned} y_t - \nu y_{xx} + y y_x &= f && \text{in } Q = (0,T)\times\Omega \\ \nu y_x(\cdot,0) &= u && \text{in } (0,T) \\ \nu y_x(\cdot,1) + \sigma y(\cdot,1) &= v && \text{in } (0,T) \\ y(0,\cdot) &= y_\circ && \text{in } \Omega = (0,1) \end{aligned}$$

$T = 1$, $\beta = 0.001$, $\nu = 0.5$, $f(t,x) = e^{-3t}\sin(2\pi x)$, $\sigma = 0.1$, $y_\circ = \sin(2\pi x)$



Numerical strategy: Update of POD basis

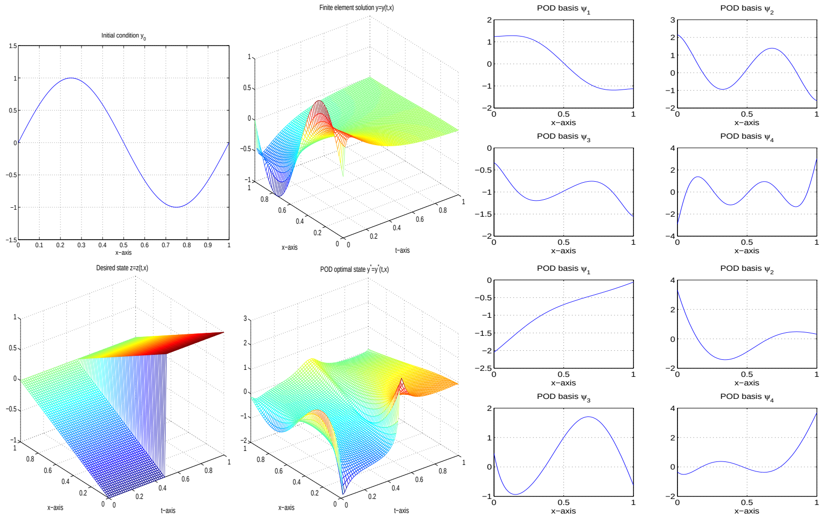
Idea: combine fast POD solver and OS-POD information

1) Choose u^0 and set $n = 0$
2) Determine PODs $\{\psi_i^n\}_{i=1}^{\ell}$ for $y^n = y^n(u^n)$
3) Solve inexactly $[\tilde{u}^n, q^n] = \mathtt{SQP}^{\ell}(m)$ (POD solver)
4) Solve dual quation for p^n (OS-POD information)
5) Compute $u^{n+1} = \tilde{u}^n - \tau\left(\mathbf{R}\tilde{u}^n(t) - \mathbf{B}(\psi^n)^T q^n(t) - \mathcal{B}^\star p^n(t)\right)$, $\tau > 0$
6) Set $n = n + 1$ and go back to 2) if $n \leq n_{\max}$

Stopping criterium: OS-POD gradient 'small', POD solver 'exact'

$\mathtt{SQP}^{\ell}(m)$: m SQP steps for the control problem with fixed POD basis

Numerical test (figures)



Numerical test (tables)

	$J(y,u,v)$
Solve for $u = v = 0$	0.22134
OS-POD	0.03813
FE-SQP	0.03765

Steps	CPU time
Generate snapshots	0.62 s
Determine POD	0.09 s
Determine ROM	0.03 s
SQP solver	14.16 s
Compute μ_i's	2.07 s
Dual FE solver	0.71 s
Backtracking in 5)	0.32 s

	$n = 0$	$n = 4$	with FE controls
$\lambda_1/\mathrm{tr}\,(\mathcal{K}_h)$	0.97187	0.87661	0.88092
$\lambda_2/\mathrm{tr}\,(\mathcal{K}_h)$	0.02209	0.08051	0.08734
$\lambda_3/\mathrm{tr}\,(\mathcal{K}_h)$	0.00579	0.02736	0.02744
$\lambda_4/\mathrm{tr}\,(\mathcal{K}_h)$	0.00025	0.00191	0.00292

Conclusions

- POD in Hilbert spaces: choice of weights and norms
  $\longrightarrow$ convergence estimates [Kunisch/V., Hinze/V., Kahlbacher/V.]
- Parameter estimation in elliptic systems
  $\longrightarrow$ Helmholtz equation [ACC Graz]
- OS-POD: update of the PODs within optimization
  $\longrightarrow$ optimal POD basis at optimal solution
  $\longrightarrow$ more complex problems
- Preprints:

  http://www.uni-graz.at/imawww/reports/index.html
- POD scriptum:

  http://www.uni-graz.at/imawww/volkwein/POD.pdf