Approximate Solution to Inverse Problems for Elliptic Equations

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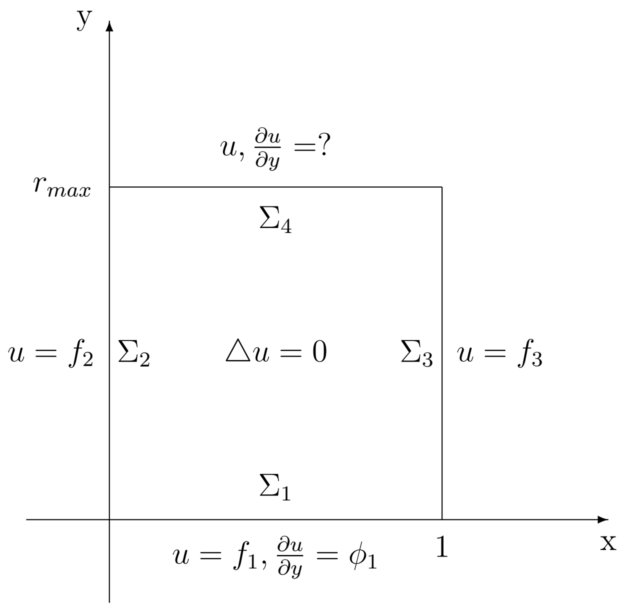
CONTENTS

- PROBLEM SETTINGS
  - Cauchy-Problem for Laplace's Equation
  - Cauchy-Problem for more general elliptic equations
  - Shape optimization
- METHOD OF LINES APPROXIMATION
- CONVERGENCE IN CASE OF DATA WITH BOUNDED FREQUENCIES
- CASE: BOUNDED SOLUTIONS
  - Stability
  - Approximability
  - Error estimates
  - Optimal Convergence
- MORE GENERAL SITUATION: $\mathrm{div}\big(a(x)\nabla u\big) = 0$
- NUMERICAL RESULTS

- **PROBLEM SETTINGS**

Cauchy–Problem
for Laplace's Equation (CPLE)



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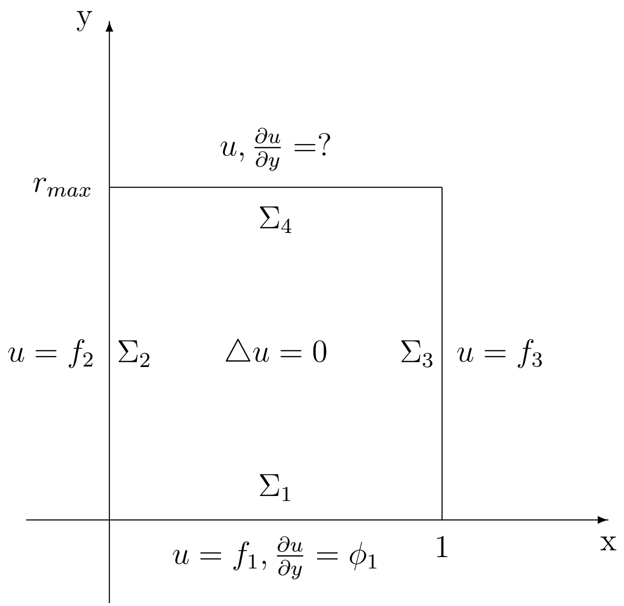


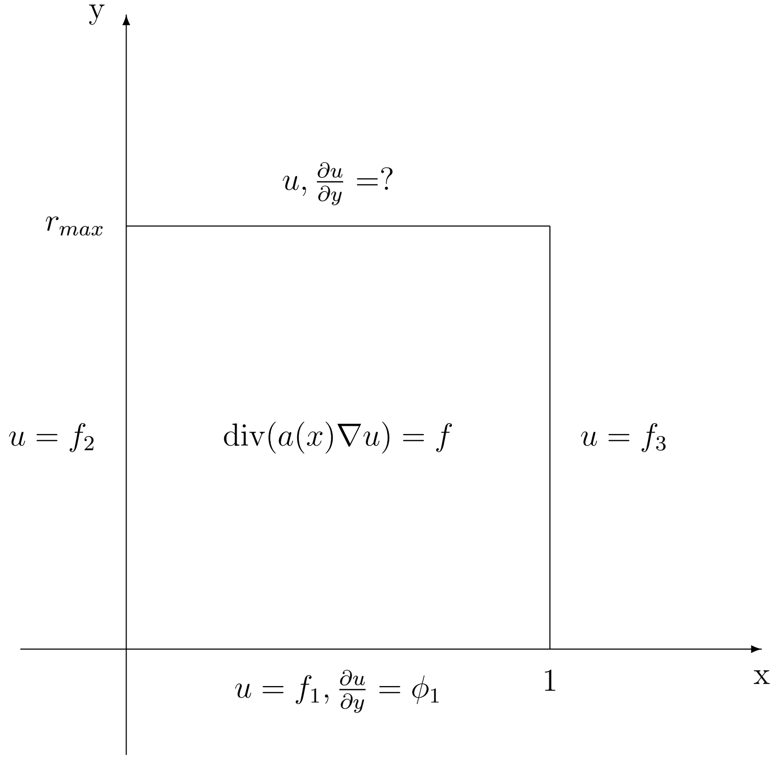
Hadamard Example (1923)

$$\Delta u = 0 \; ,$$
$$u(x,0) = 0 \; ,$$
$$\frac{\partial u}{\partial y}(x,0) = \frac{1}{n}\sin(n\pi x) =: g_n \; , \; x \in (0,1)$$
$$u(0,y) = u(1,y) = 0 \; , \; 0 \le y \le 1$$
$$\Longrightarrow \; u_n(x,y) = (n\pi)^{-2}\sin(n\pi x)\sinh(n\pi y) \; , \quad (x,y) \in [0,1]\times[0,1]$$

Illposed: $g_n \to 0$, but $u_n(x,y) \to \infty$ $(n \to \infty)$ for any $y > 0$

Lit.: Lavrentiev ('56), Payne ('60ff), Han ('82), Falk ('90), M. Kubo ('94), Kabanikhin + Karchevsky ('95), Fayazov + Lavrentiev ('95) and many others.

Cauchy–Problem for more general elliptic equations



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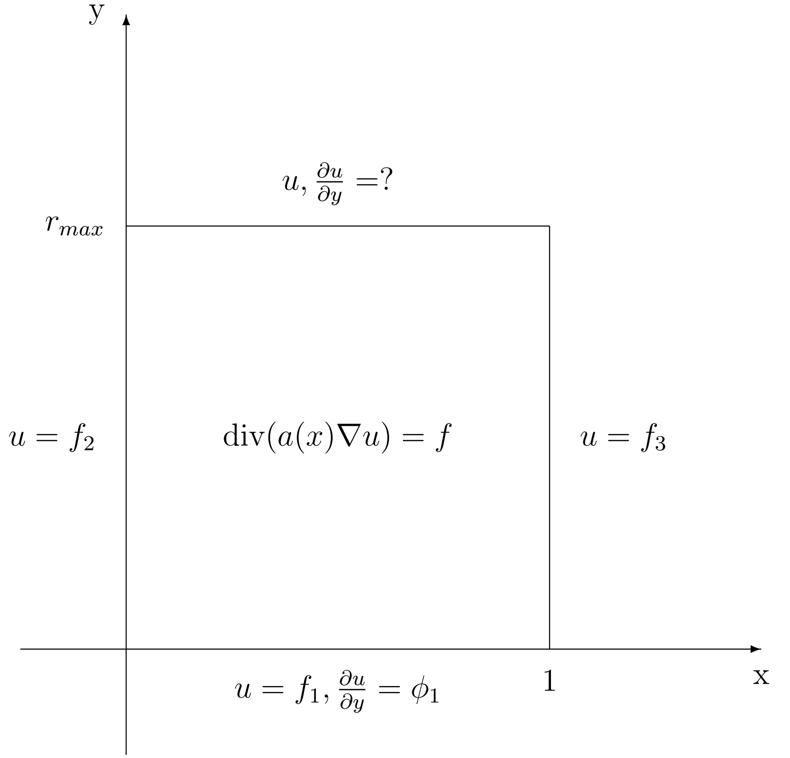
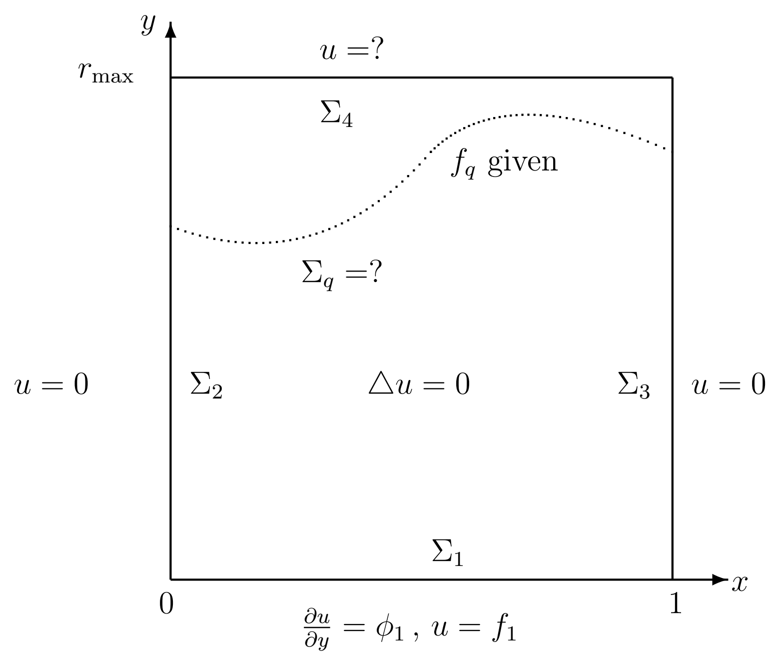


Shape Optimization Problem



Remarks

- All three problems are illposed.
- f, f_1, f_2, f_3 can be set to zero in the Cauchy-Problem for more general elliptic equations. Note: $a = a(x)$.
- The shape optimization problem needs the solution of the Cauchy–Problem beforehand.

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Applications

- Medicine: Electrocardiology
- Geology: Gravimetric search of resources
- Steel production: Thickness of furnace wall
- Stationary Inverse Heat Conduction Problems

- **METHOD OF LINES APPROXIMATION**

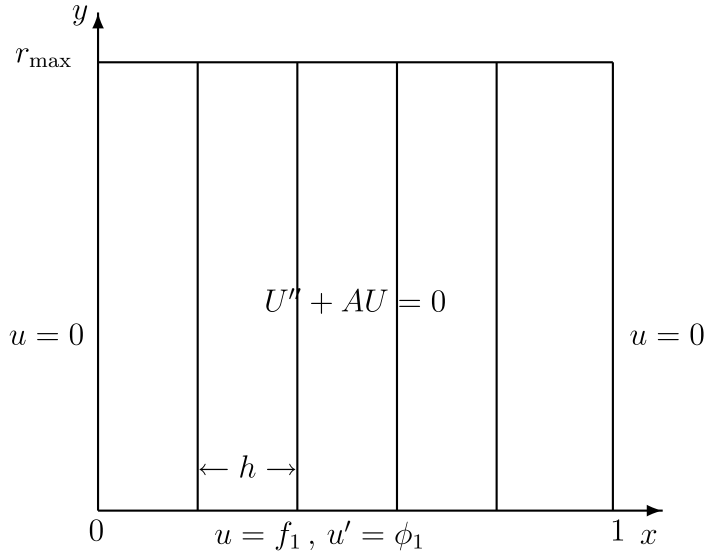
$x_i = ih\,,\; i = 0, \ldots, N$
$\triangle_h u(x_i, y) = \frac{\partial^2 u}{\partial y^2}(x_i, y) + \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}$
$u(x_i, .) \approx u_i\,,\; \frac{\partial^2 u}{\partial y^2}(x_i, .) \approx u_i''\,,$
$U = (u_1, \ldots, u_{N-1})$
$\mathbf{\triangle_h u = 0 \Longleftrightarrow U'' + AU = 0}$

with

$$A := \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & -2 & 1 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & -2 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & -2 \end{pmatrix}$$

$\in \mathbb{R}^{N-1,N-1}$

Boundary Conditions:
$u_i(0) = f_1(x_i)\,,\; u_i'(0) = \phi_1(x_i)\,,$
$i = 1, \ldots, N-1$



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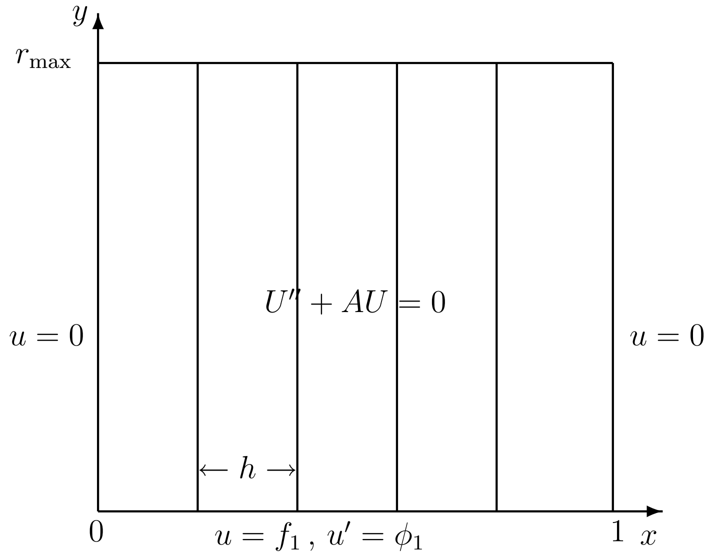

The above system can be decoupled, since eigenvalues and eigenvectors of A are known.

$$
\begin{aligned}
& U'' + AU = 0 \\
\iff\ & WU'' + WAU = 0 \\
\iff\ & WU'' + \underbrace{WAW^{-1}}_{=D} WU = 0 \\
\iff\ & (WU)'' + D(WU) = 0 \\
\overset{V:=WU}{\iff}\ & V'' + DV = 0\,,\ V = (v_1, \dots, v_{N-1})^\top \\
\iff\ & v_i'' + \lambda_i v_i = 0\,,\ i = 1, \dots, N-1
\end{aligned}
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$\lambda_i =$ eigenvalues , $D =$ diag (λ_i) ,
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$$
\Longrightarrow \quad v_k(y) = \xi_k \exp(\sqrt{-\lambda_k} y) + \eta_k \exp(-\sqrt{-\lambda_k} y)\,,\ k = 1, \dots, N-1
$$

$$
\Longrightarrow \xi_k = \sqrt{\frac{h}{2}} \sum_{j=1}^{N-1} \left(\sin(kjh\pi) f_1(x_j) + \frac{h}{2} \frac{\sin(khj\pi)}{\sin\left(kh\frac{\pi}{2}\right)} \phi_1(x_j) \right)
$$

$$
\eta_k = \sqrt{\frac{h}{2}} \sum_{j=1}^{N-1} \left(\sin(kjh\pi) f_1(x_j) - \frac{h}{2} \frac{\sin(khj\pi)}{\sin\left(kh\frac{\pi}{2}\right)} \phi_1(x_j) \right)
$$

$\Longrightarrow$ solution $U = (U_1, \ldots, U_{N-1})$ of $U'' + AU = 0$:

$$
\begin{aligned}
u_i(y) &= (WV)_i(y) \\
&= 2h \cdot \sum_{k=1}^{N-1} \left(\sin(ikh\pi) \left(\cosh(\sqrt{-\lambda_k} y) \sum_{j=1}^{N-1} \sin(kjh\pi) f_1(x_j) \right.\right. \\
&\qquad \left.\left. + \frac{h}{2\sin\left(kh\frac{\pi}{2}\right)} \sinh(\sqrt{-\lambda_k} y) \sum_{j=1}^{N-1} \sin(kjh\pi) \phi_1(x_j) \right) \right)
\end{aligned}
$$

Remarks:

1) The CPLG is solved by $u(x,y) = \sum_{k=1}^{\infty} g_k(x,y)$ with

$$g_k(x,y) \;=\; 2\sin(k\pi x)\left((f_1(.), \sin(k\pi .))_{L_2} \cosh(k\pi y) + \frac{(\phi_1(.), \sin(k\pi .))_{L_2}}{k\pi} \sinh(k\pi y) \right)$$

provided the series converges.

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2) The method of lines approximation is still illposed on every line.

3) The CPLE is conditionally well-posed if data $f_1, \phi_1 \in D_M$ (data with *bounded frequencies*), where

$$D_M \;=\; \left\{\phi \in C^1(0,1) \,\middle|\, \phi(0) = \phi(1) = 0\,,\; \int_0^1 sin(k\pi t)\phi(t)\,dt = 0,\; k > M\right\}$$

Solution:

$$u(x,y) \;=\; \sum_{k=1}^{M}\left(2\sin(k\pi x)\left((f_1(.),\sin(k\pi .))_{L_2}\cosh(k\pi y) + \frac{(\phi_1(.),\sin(k\pi .))_{L_2}}{k\pi}\sinh(k\pi y)\right)\right)$$

4) Data spaces

$$D : \quad = \left\{\phi \in C^1[0,1] \,\middle|\, \phi(0) = \phi(1) = 0\right\}$$

$$D_M : = \left\{\phi \in D \,\middle|\, \int_0^1 \sin(k\pi t)\phi(t)\,dt = 0, k > M\right\}$$

$$D_M^h : = \left\{\Phi \in \mathbb{R}^{N-1} \,\middle|\, \sum_{j=1}^{n-1} \sin(k\pi jh)\Phi_j = 0,\ N > k > M\right\} \quad (N > M)$$

where $\Phi := (\phi(h), \ldots, \phi\big((N-1)h\big)^\top,\ \phi \in D.$

Projection $P_M : \mathbb{R}^{N-1} \longrightarrow D_M^h$

$$(P_M\Phi)_j = \sum_{k=1}^{M} \left(2h \sum_{\ell=1}^{N-1} \Phi_\ell \sin(k\pi\ell h)\right) \sin(k\pi jh)$$

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$\Longrightarrow$ Fourier coeff. of $\phi \in D_M$ are finite sums,

$$\int_0^1 \phi(t)\sin(k\pi t)\,dt \; = \; h \sum_{j=1}^{N-1} \sin(k\pi jh)\Phi_j$$

$\forall\ M, N, k \in \mathbb{N},\ M < N, k < N, f \in D_M\,.$

- **CONVERGENCE AND ERROR ESTIMATES**
  (in case of data with bounded frequencies; assume $N > M(h = 1/M)$):

  If $f_1, \phi_1 \in D_M$ then $|u(x_i, y) - u_i(y)| = O(h^2) \quad (h \to 0)$.

  For perturbed data $\|f_1 - f_1^\varepsilon\|_\infty = O(\varepsilon), \|\phi_1 - \phi_1^\varepsilon\|_\infty = O(\varepsilon)$:

$$
\begin{aligned}
\left|u(x_i, y) - u_{i,\varepsilon}^*(y)\right| &\leq \frac{M^4\pi^3 y}{12} \exp(M\pi y)(\|f_1\|_{L_1} + \|\phi_1\|_{L_1})h^2 + \frac{4M^2(M+1)}{\sqrt{2}} \exp(\pi M y)\varepsilon \\
&= O(h^2 + \varepsilon) \, ,
\end{aligned}
$$

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  where $u_{i,\varepsilon}^*$ is the semidiscrete solution belonging to data, which are perturbed and projected onto D_M afterwards, i.e. $F_{1,\varepsilon}^* = P_M F_{1,\varepsilon}$, $\Phi_{1,\varepsilon}^* = P_M \Phi_{1,\varepsilon}$,

$$\begin{aligned} u_{i,\varepsilon}^* &= 2h \sum_{k=1}^{M} \sin(ikh\pi) \cosh(\sqrt{-\lambda_k} y) \sum_{j=1}^{N-1} \sin(kjh\pi)(F_{1,\varepsilon}^*)_j \\ &+ h^2 \sum_{k=1}^{M} \left(\frac{\sin(ikh\pi)}{\sin(kh\frac{\pi}{2}} \sinh(\sqrt{-\lambda_k} y) \sum_{j=1}^{N-1} \sin(kjh\pi)(\Phi_{1,\varepsilon}^*)_j \right) . \end{aligned}$$

- **CASE: BOUNDED SOLUTION** (on Σ_4)

Stability Theorem: If $u \in C^2\,(\mathrm{int}(\Omega)) \cap C\,(\Omega)$ s.t.

$$\begin{aligned}
\triangle u(x,y) &= 0 \quad \text{in} \quad \mathrm{int}(\Omega) \\
u &= 0 \qquad \text{on} \quad \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \; (i.e.\ f_1 = f_2 = f_3 = 0) \\
\frac{\partial u}{\partial y}(x) &= \phi_1(x) \quad \text{on} \quad \Sigma_1 \\
\|u\|_{L_2(\Sigma_4)} &\leq E\,. \qquad\qquad (*)
\end{aligned}$$

Then

$$\|u(.,y)\|_{L_2} \;\leq\; R_1 \|\phi_1\|_{L_2}^{1-\frac{y}{r_{max}}} E^{\frac{y}{r_{max}}}$$

for all $y \in [0, r_{max}]$, with $R_1 = \max(r_{\max}, 1)$.

Remarks:

1) The proof uses *logarithmic convexity* of $s(y) = \left\|u(\cdot,y)\right\|_{L_2}^2 / y^2$, i.e. $ln(s)$ convex.

2) CPLE is conditionally wellposed in this case.

Approximability:

Projection (orth.) $P_M : D \longrightarrow D_M$ satisfies

$$\|\phi_1 - P_M\phi_1\|_{L_2(\Sigma_1)} \leq \frac{E}{r_{max}^2(1-\exp(-4\pi r_{max}))} \cdot \frac{M}{\exp(M\pi r_{max})},$$

provided

$$(*) : \|u\|_{L_2(\Sigma_u)} \leq E,$$

holds.

Assume $f_1 = 0$, $\|\phi_1 - \phi_1^\varepsilon\|_\infty = 0(\varepsilon)$. Then ...

Error estimates:

$$\|(u - \overline{(u_\varepsilon^*)_h})(.,y)\|_{L_2}$$

$$\leq\ C_1(r_{max})E \cdot \left(\frac{M}{\exp(M\pi r_{max})}\right)^{1-\frac{y}{r_{max}}}$$

Error estimates:

$$
\begin{aligned}
&\|(u - \overline{(u^*_\varepsilon)_h})(.,y)\|_{L_2} \\
&\quad \le\ C_1(r_{max}) E \cdot \left(\frac{M}{\exp(M\pi r_{max})}\right)^{1-\frac{y}{r_{max}}} \\
&\quad +\ C_2(y) \frac{2}{\pi^2} \frac{\sinh(M\pi y)}{M} \varepsilon \\
&\quad +\ \frac{M^4 \pi^3 y}{12} \exp(M\pi y) \|(\phi_1)^*_\varepsilon\|_{L_1} h^2 \quad \forall y \in [0, r_{\max}) \,.
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{(u^*_\varepsilon)_h}(x,y) &= \sum_{k=1}^{M} \left(2h \sum_{j=1}^{N-1} \sin(k\pi jh) u^*_{j,\varepsilon}(y)\right) \sin(k\pi x) \\
&= \text{continuation of } \left(u^*_{1,\varepsilon}(y), \ldots, u^*_{N-1,\varepsilon}(y)\right)^\top \text{ in } D_M \\
u^*_{i,\varepsilon}(y) &= \text{solution on i-th line with data } P_M \phi_1^\varepsilon \,.
\end{aligned}
$$

Error estimates:

$$\|(u - \overline{(u^*_\varepsilon)_h})(.,y)\|_{L_2}$$

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$$+ C_2(y)\frac{2}{\pi^2}\frac{\sinh(M\pi y)}{M}\varepsilon$$

$$+ \frac{M^4\pi^3 y}{12}\exp(M\pi y)\|(\phi_1)^*_\varepsilon\|_{L_1}h^2 \quad \forall y \in [0, r_{\max})\,.$$

where

$$\overline{(u^*_\varepsilon)_h}(x,y) = \sum_{k=1}^{M}\left(2h\sum_{j=1}^{N-1}\sin(k\pi jh)u^*_{j,\varepsilon}(y)\right)\sin(k\pi x)$$

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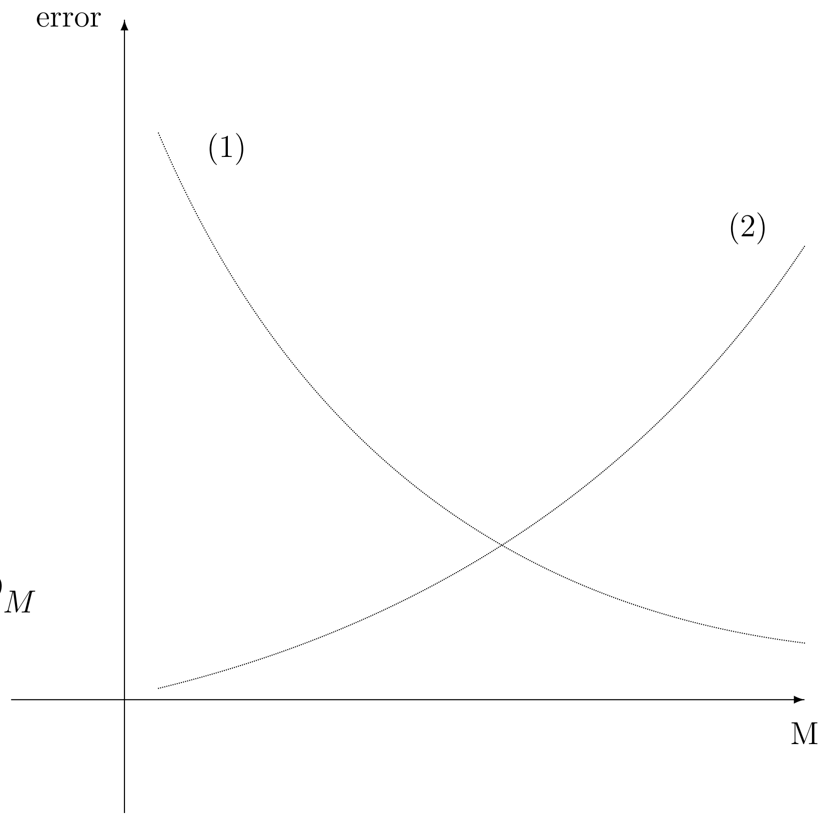

(1) projection error, (2) data error; ε, h fixed

Optimal Convergence:

If $M = \left\lceil \frac{\ln 1/\varepsilon}{\pi r_{\max}} \right\rceil$ and $h = \sqrt{\varepsilon}$, then

$$\begin{aligned}
\|(u - \overline{(u_\varepsilon^*)_h})(.,y)\|_{L_2} \leq\ & C_1(r_{max}) E \cdot \left(\frac{\varepsilon \cdot \ln\left(\frac{1}{\varepsilon}\right)}{\pi r_{max}} + \varepsilon \right)^{1-\frac{y}{r_{max}}} \\
+\ & 2 C_2(y) \exp(\pi y) r_{max} \frac{\varepsilon^{1-\frac{y}{r_{max}}}}{\ln\left(\frac{1}{\varepsilon}\right)} \\
+\ & \frac{y}{12\pi} \|(\phi_1)_\varepsilon^*\|_{L_1} \frac{\left(\ln\left(\frac{1}{\varepsilon}\right) + \pi r_{max}\right)^4 \cdot \exp(\pi y) \cdot \varepsilon^{1-\frac{y}{r_{max}}}}{r_{max}^4} \quad \longrightarrow 0 \ (\varepsilon \to 0)
\end{aligned}$$

- **MORE GENERAL SITUATION:** $div\big(a(x)\nabla u\big) = 0$
  (Assumptions: $0 < r_a \leq a(x) \leq R_a, \big|a'(x)\big| \leq R'_a$)

  Same program:

- Method of lines approximation
  Difficulty: Eigenvalues, -vectors are not explicitly known
- Analyse discrete Sturm-Liouville-Eigenvalue Problem
  (convergence of eigenvalues and eigenvectors)
- Use logarithmic convexity etc.

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  (convergence of eigenvalues and eigenvectors)

  – Use logarithmic convexity etc.

  Similar results as for CPLE w.r.t. $(v_h, w_h)_{0,a} = h \sum_{j\in\mathbb{Z}} a(x_j) v_h(x_j) w_h(x_j)$;

  use projection P_M onto D_M also w.r.t. $(\cdot,\cdot)_{0,a}$

Convergence: For $M = \left\lceil \ln(1/\varepsilon)/(\pi r_{\max}\sqrt{\frac{R_a}{r_a}}) \right\rceil$, $h = \sqrt{\varepsilon}$

$$
\begin{aligned}
\|u - \overline{u^*_{\varepsilon,h}}\|_a \;&\leq\; CE\left(\frac{\varepsilon^{\frac{r_a}{R_a}}\ln\left(\frac{1}{\varepsilon}\right)}{\pi r_{max}\sqrt{\frac{R_a}{r_a}}} + \varepsilon^{\frac{r_a}{R_a}}\right)^{1-\frac{y}{r_{max}}} \\
&+\; \frac{R_a^2 r_{max} C(y)\exp\left(\sqrt{\frac{R_a}{r_a}}\pi y\right)}{r_a}\cdot\frac{\varepsilon^{1-\frac{y}{r_{max}}}}{\ln\left(\frac{1}{\varepsilon}\right)} \\
&+\; C\varepsilon^{1-\frac{y}{r_{max}}}\ln\left(\frac{1}{\varepsilon}\right) \longrightarrow 0\,(\varepsilon \to 0)
\end{aligned}
$$

- **Numerical results for Cauchy Problem (Hadamard's example):** $a = \text{const.}$

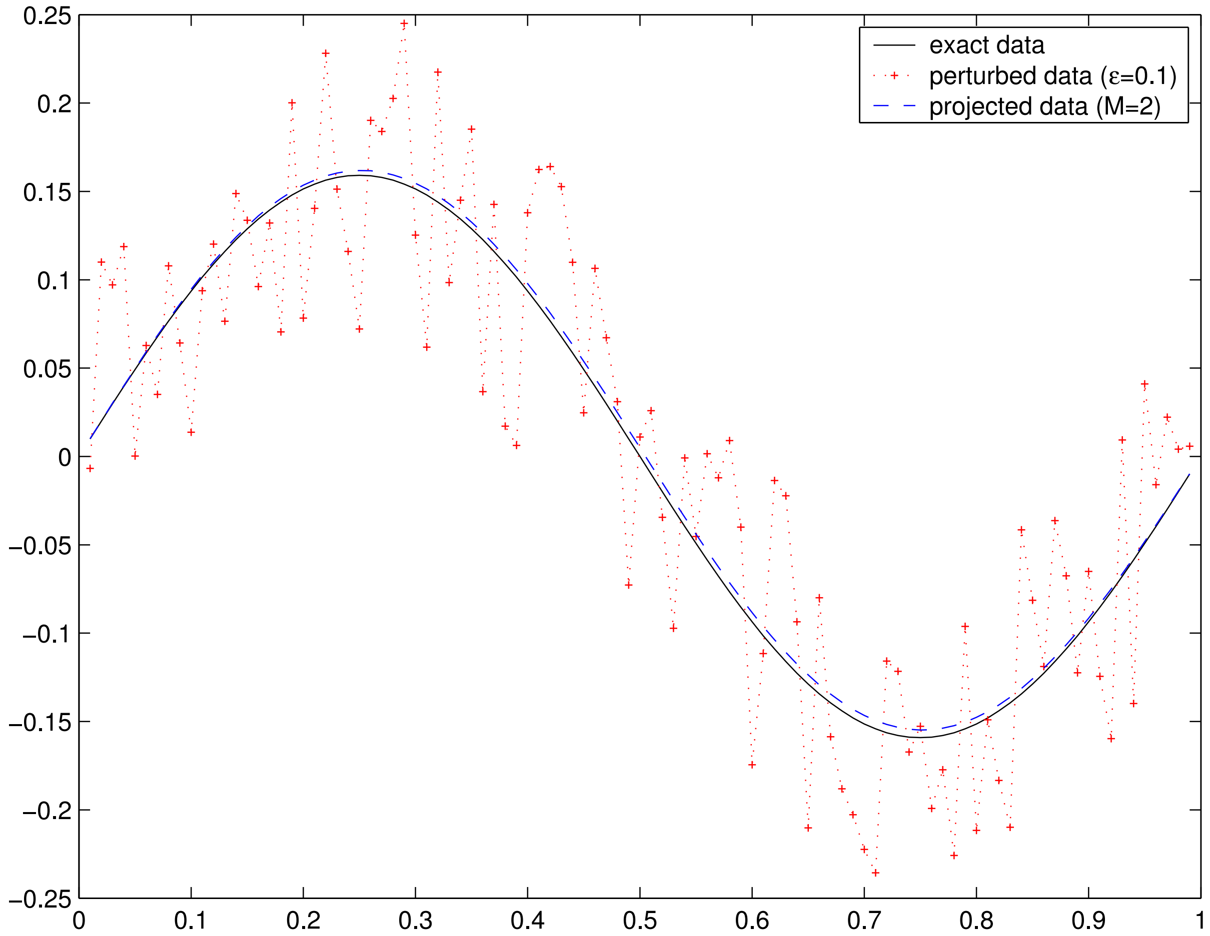

Figure 1: Projected data Φ_1 (from Hadamard Example) at $y = 0$ onto D_M with $\varepsilon = 0.1, h = \frac{1}{100}$, $m = 2$, $M = 2$

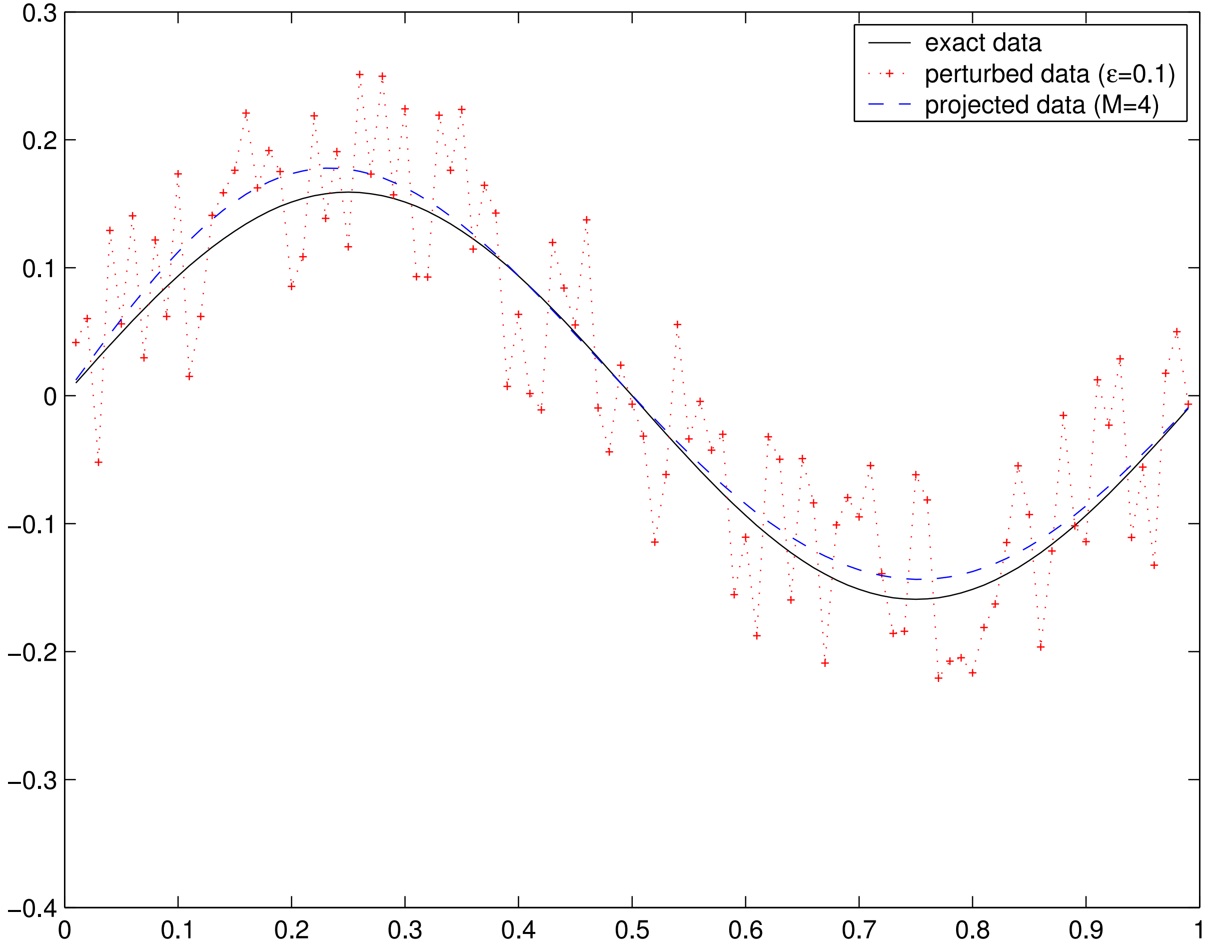

Figure 2: same as Fig. 1 with $M = 4$

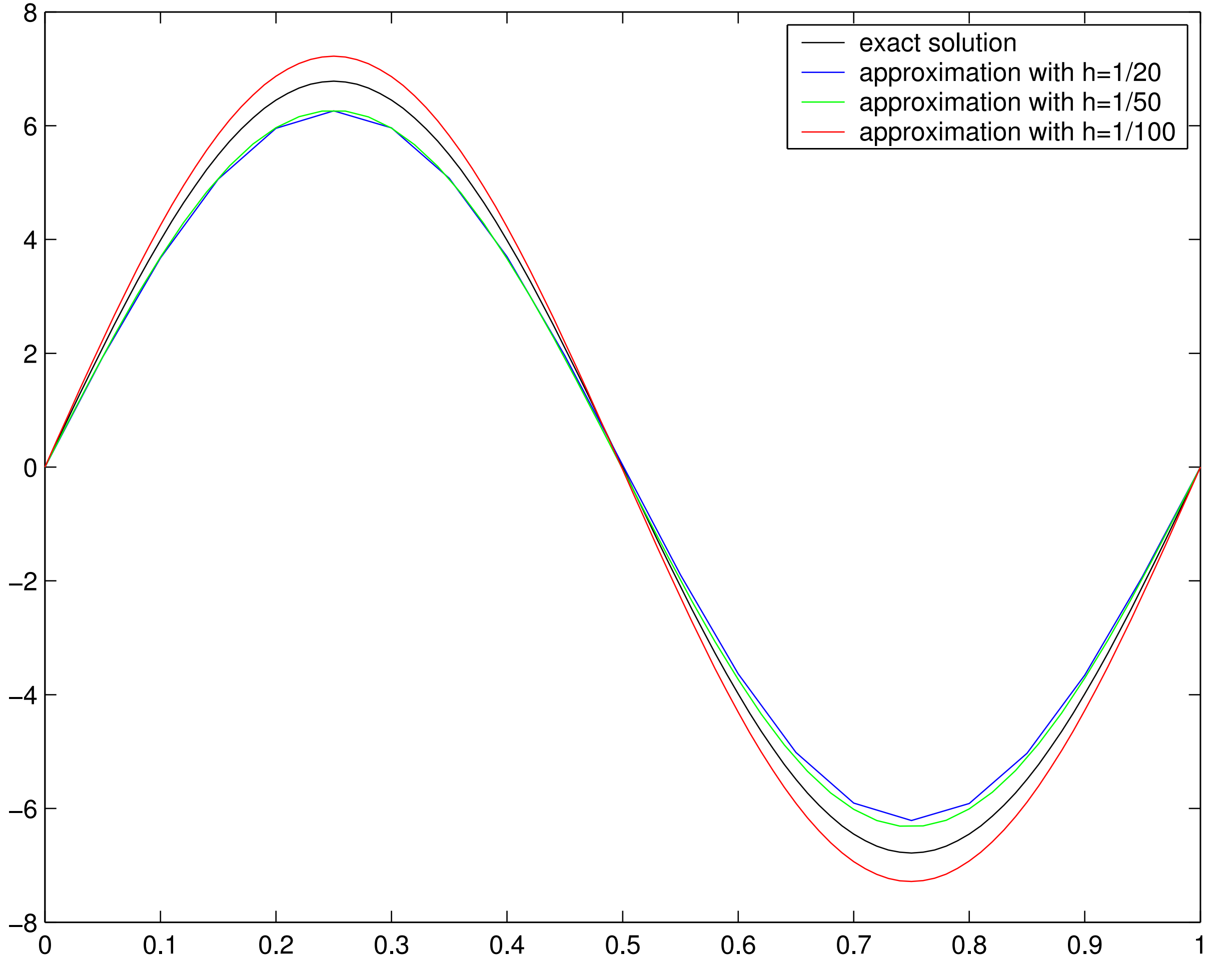

Figure 3: Solution and line method approximations of Hadamard Example at $y = 1$ for $\varepsilon = 0.1$, $m = 2$, $M = 2$ and different h's

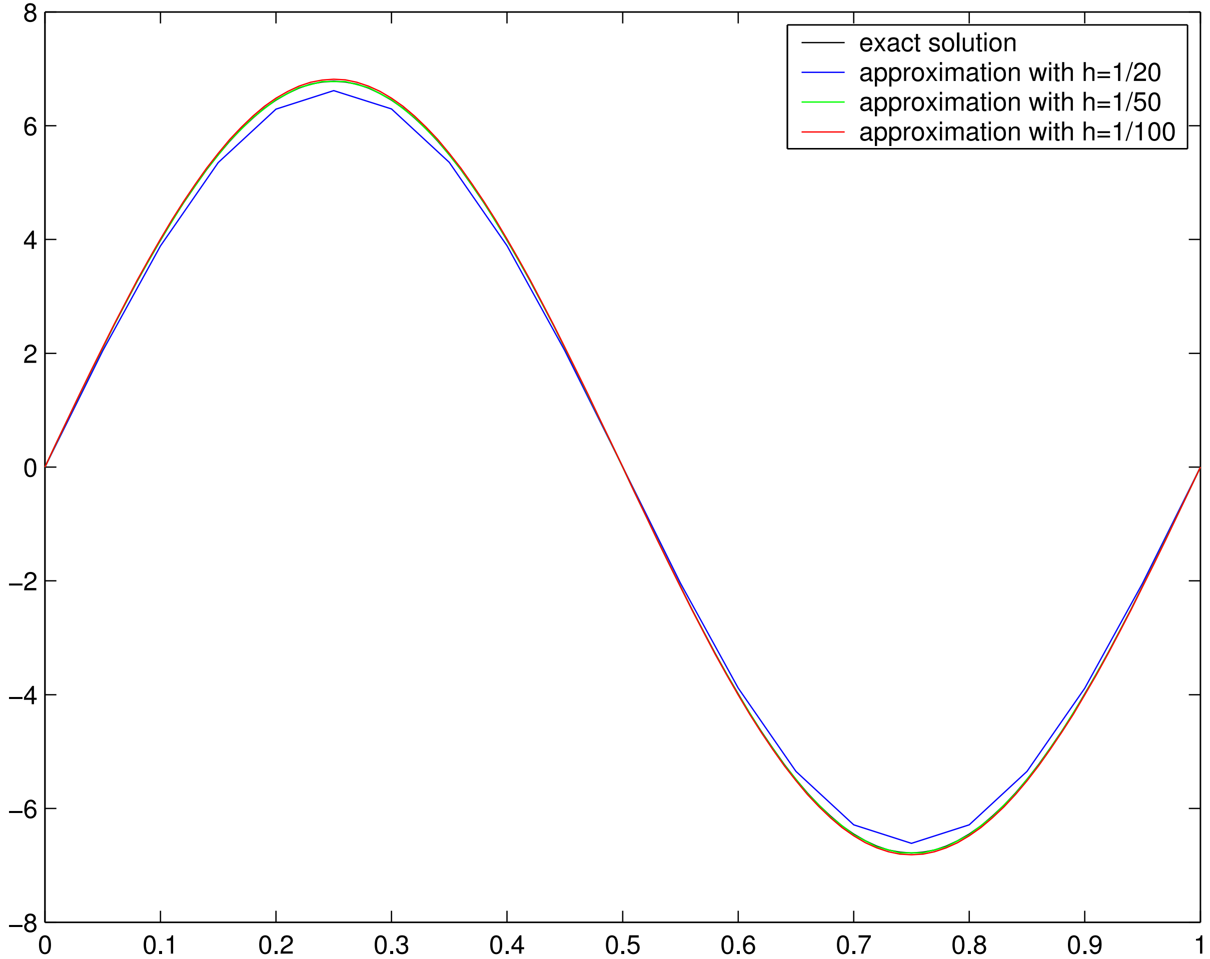

Figure 4: same as Fig. 3 with $\varepsilon = 10^{-2}$

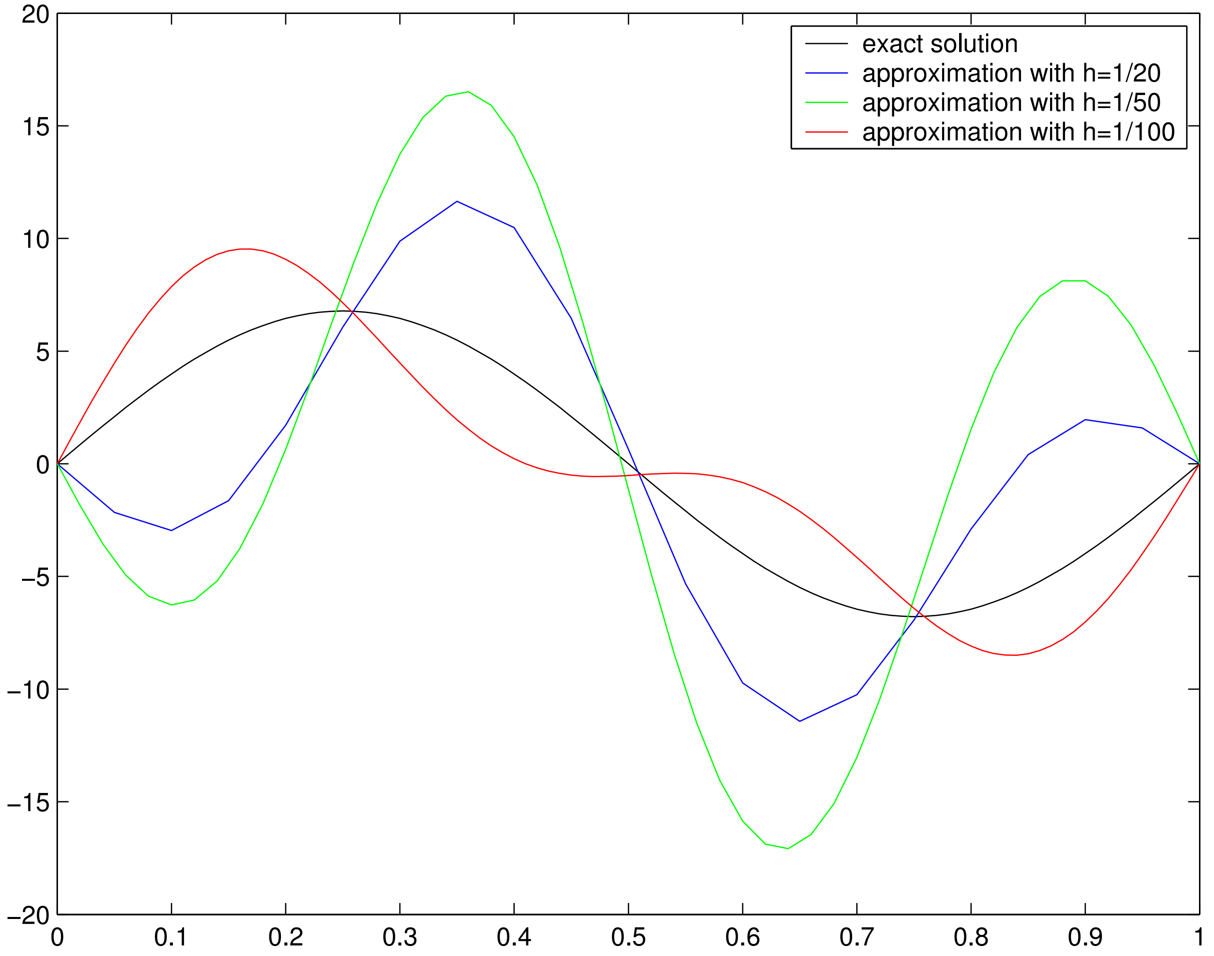

Figure 5: Solution and line method approximations of Hadamard Example at $y = 1$ **for** $\varepsilon = 10^{-2}$, $m = 2$, $M = 4$ **and different** h**'s**

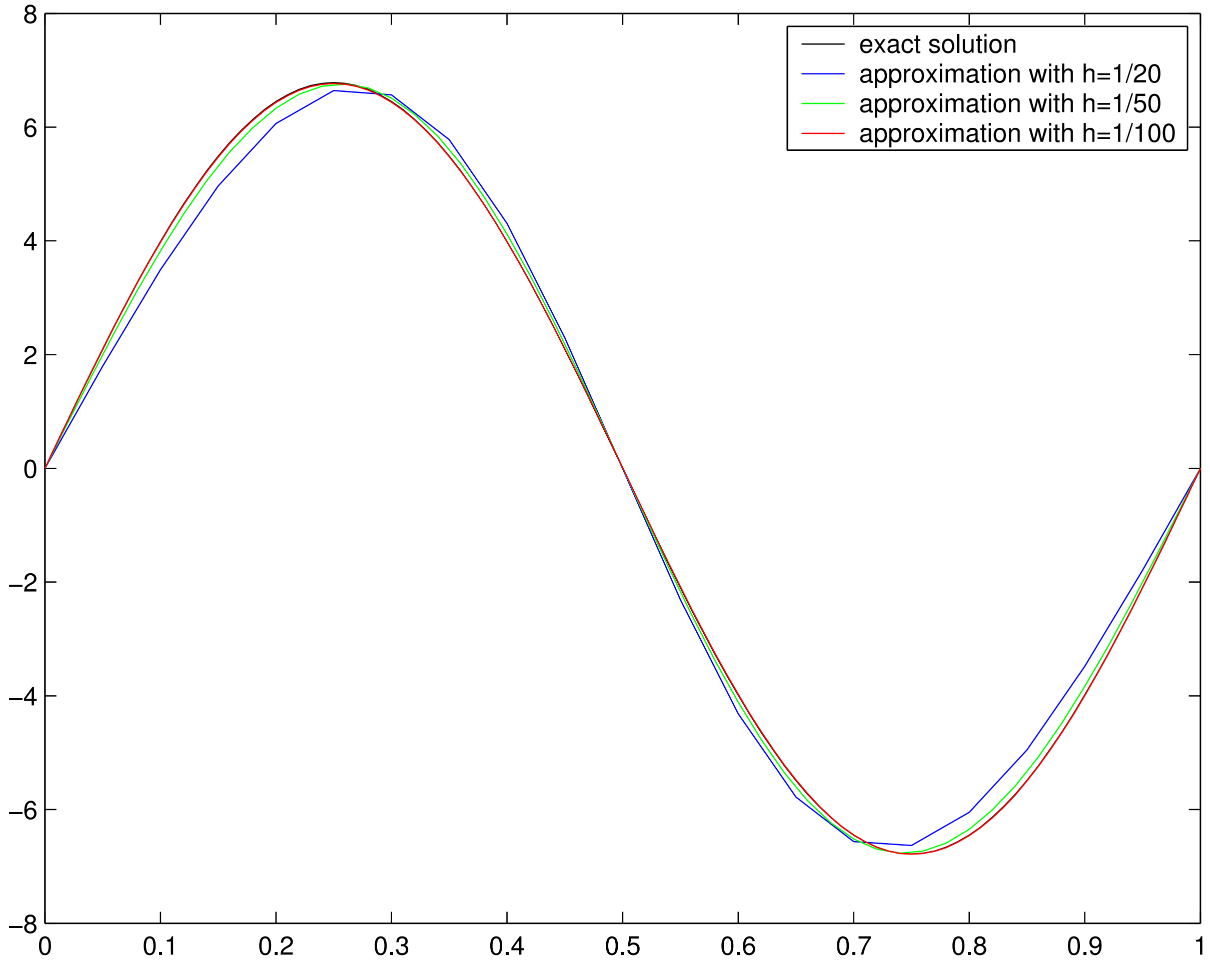

Figure 6: same as Fig. 5 with $\varepsilon = 10^{-4}$ (Note: $\varepsilon \approx 1/\exp$ M)

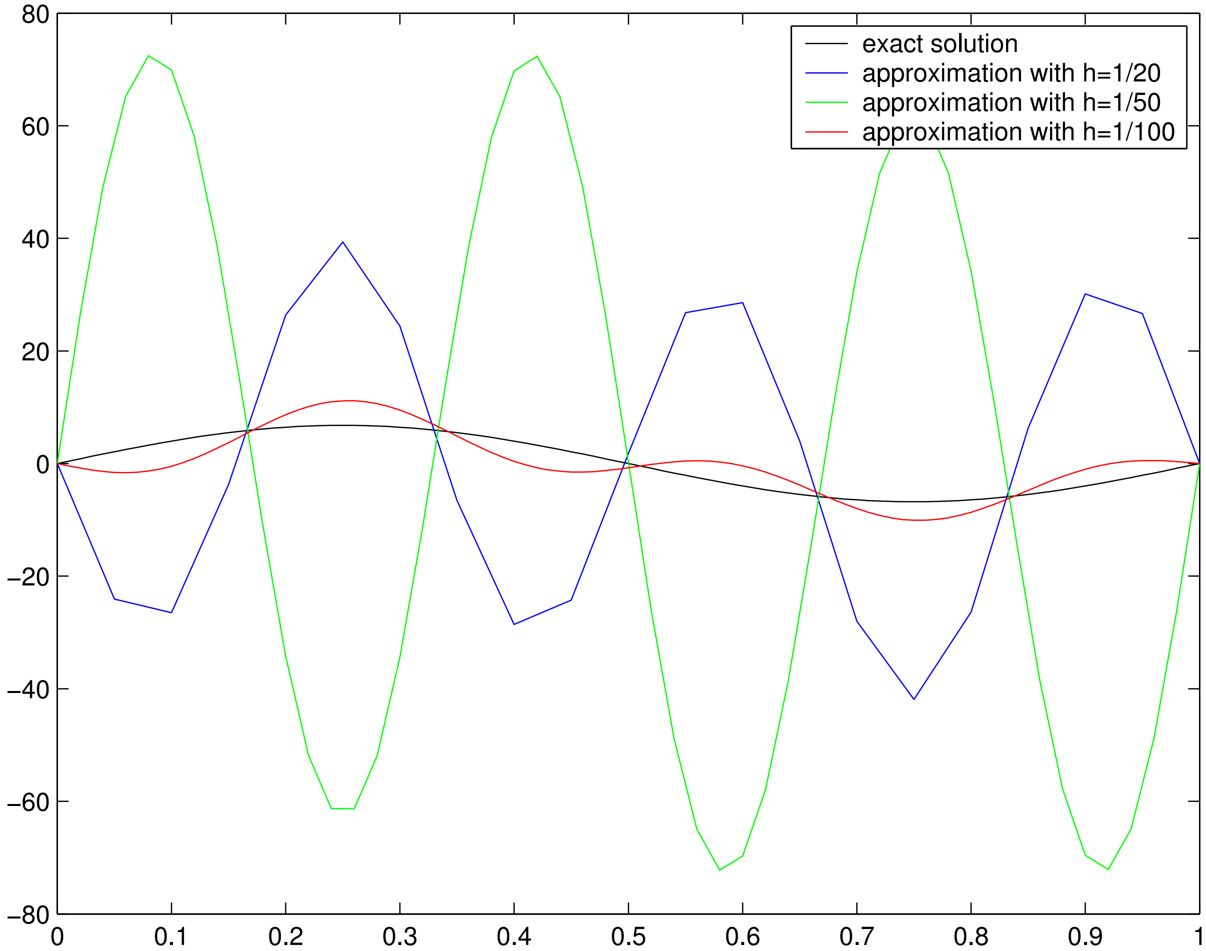

Figure 7: Solution and line method approximations of Hadamard Example at $y = 1$ for $\varepsilon = 10^{-4}$, $m = 2$, $M = 6$ and different h's

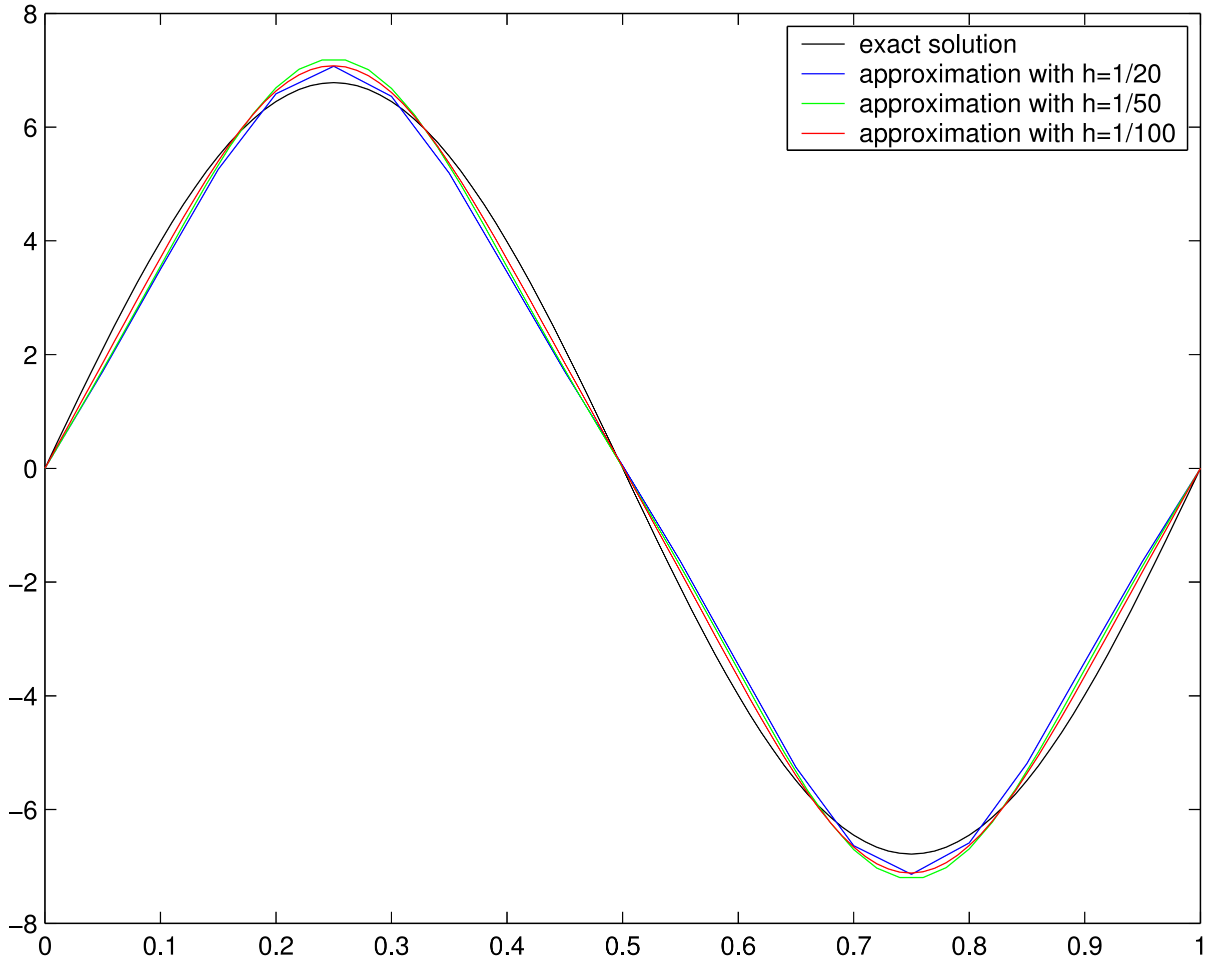

Figure 8: same as Fig. 7 with $\varepsilon = 10^{-6}$

• **Numerical results for Cauchy-Problem:** $a(x) = x + 1$

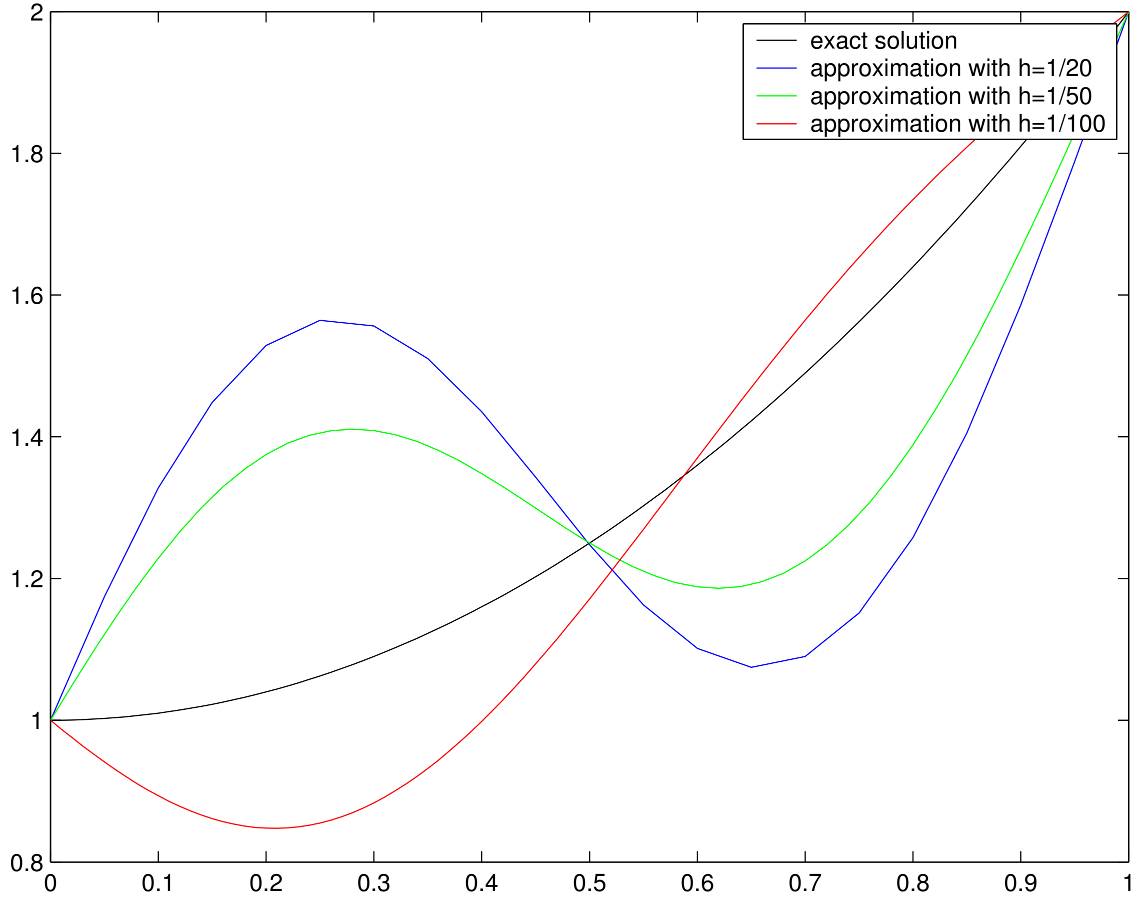

Figure 9: Solution $u = x^2 + y^2$ **and line method approximations in case of** $a(x) = x + 1$ **at** $y = 1$ **for** $\varepsilon = 10^{-1}$, $M = 2$**, different** h**'s (and** $\bar{h} = \frac{1}{200}$**)**

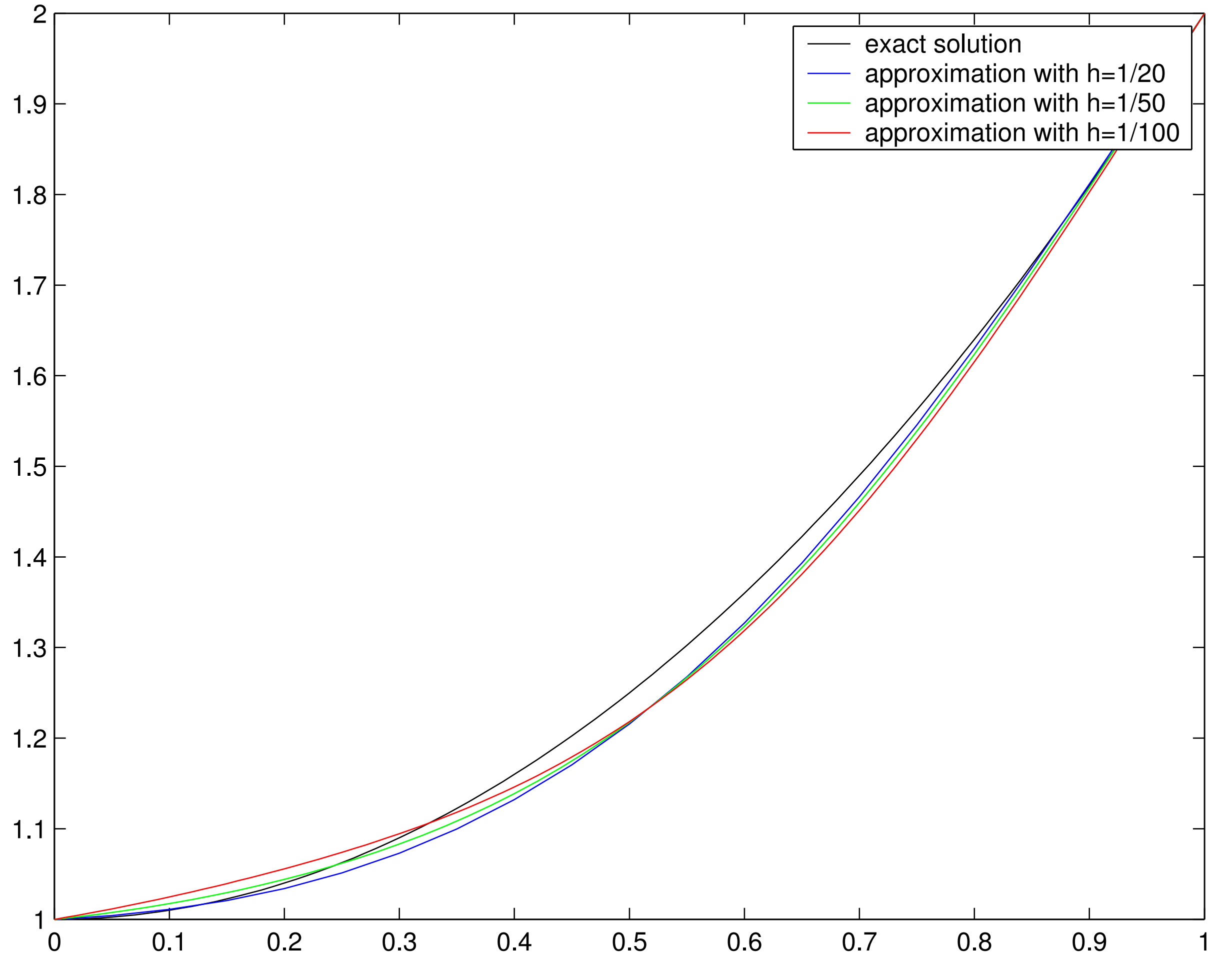

Figure 10: same as Fig. 9 with $\varepsilon = 10^{-3}$

Conclusions for Cauchy problem:

When the 2-d domain is a rectangle, or can be transformed to a rectangle, then

- the method of lines is a computable, efficient and convergent approximation scheme;
- the regularization parameter M (i.e. dimension of data space) can be chosen – and computed – in an optimal way depending on the magnitude ε of data perturbations, the bound E on the unknown part of the boundary, and the mesh width h.
- The general case $\nabla(a(x)\nabla u) = 0$ can be treated similarly.

- **Numerical results for shape optimization**

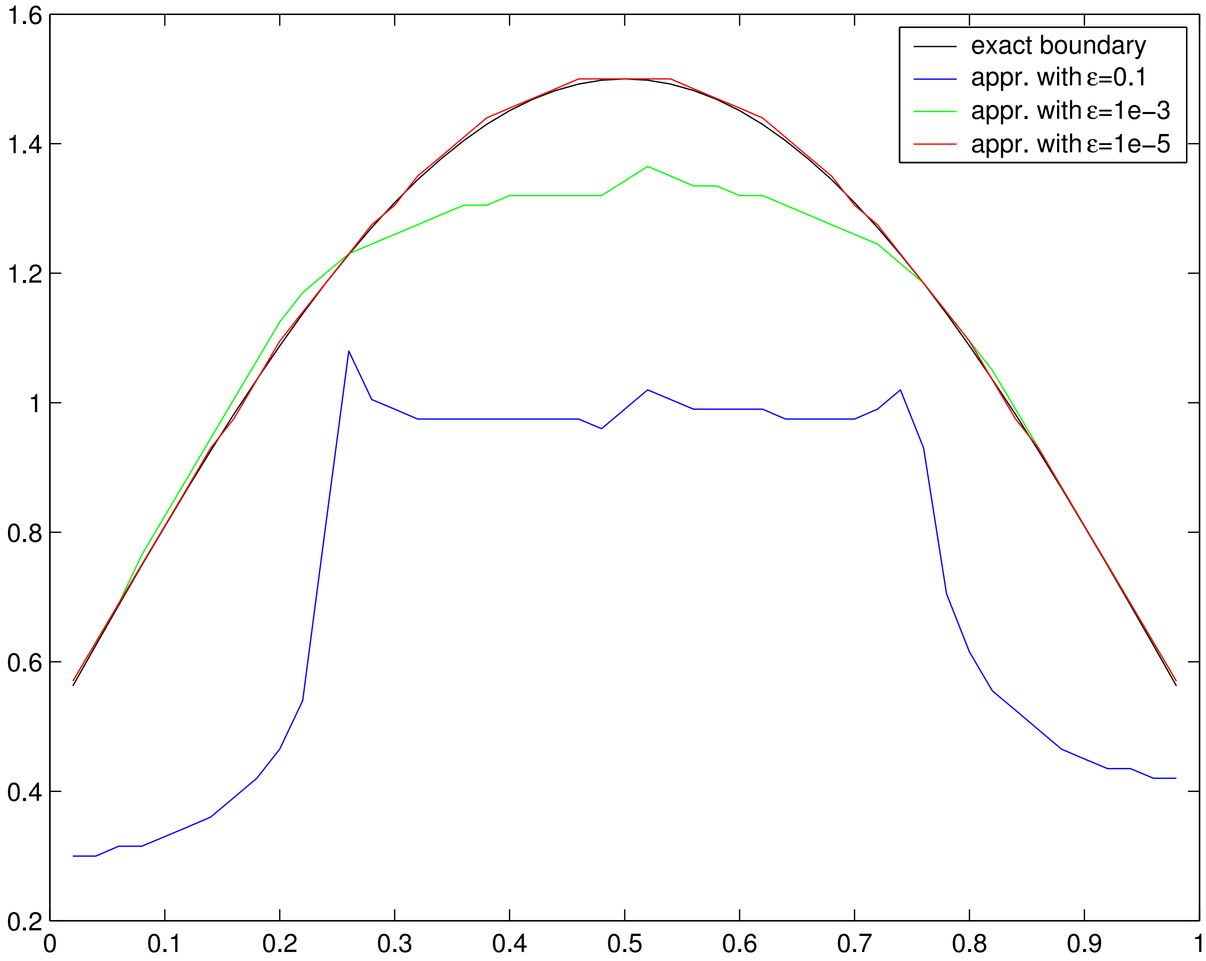

Figure 11: Solution of Hadamard example and approximation of shape identification with $m = 2,\ M = 4,\ h = \frac{1}{50}$ and different ε's (exact form of shape $\frac{1}{2} + \sin \pi x$)

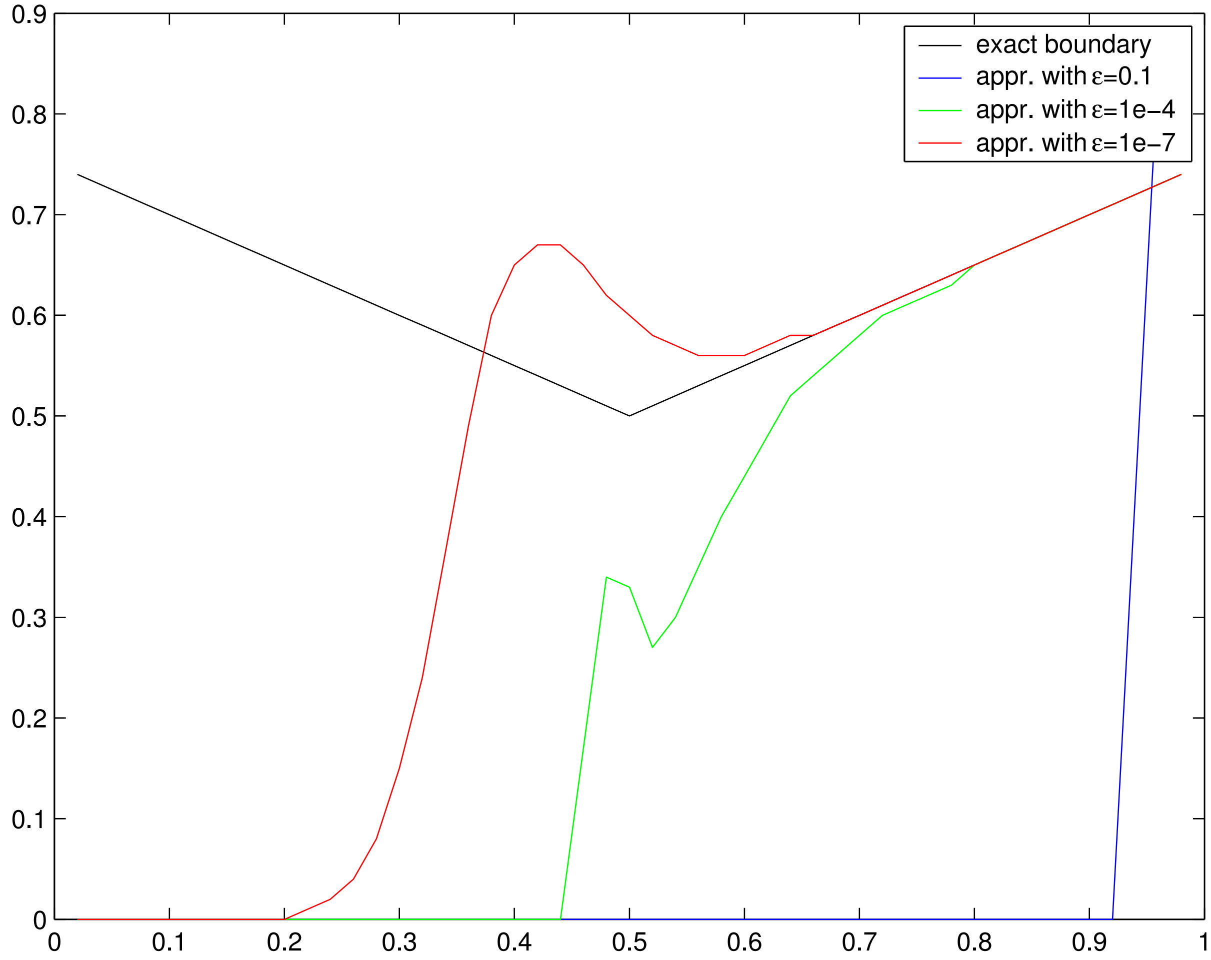

Figure 12: same as Fig. 11 (solution $u = x^{10}y^{10}$, exact shape $\frac{1}{2}\left(1 + |\frac{1}{2} - x|\right)$)