Representations of p-adic groups

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Abstract

These are expanded notes from my talks at the Current Developments in Mathematics conference in March 2022. They give an introduction to what p-adic groups are, present key techniques used in the study of representations of p-adic groups and conclude with an overview of recent developments.

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1 Introduction

A fundamental problem in representation theory is the construction of all (irreducible, smooth, complex) representations of certain matrix groups, called *p-adic groups* (defined in Section 2), which include groups like the general linear group ($\mathrm{GL}_n(F)$), the group of $n \times n$ matrices of determinant one ($\mathrm{SL}_n(F)$), the subgroup of $\mathrm{SL}_n(F)$ consisting of matrices that preserve an inner product ($\mathrm{SO}_n(F)$) and the subgroup of $\mathrm{SL}_{2n}(F)$ consisting of matrices that preserve a symplectic form ($\mathrm{Sp}_{2n}(F)$). Here F denotes a non-archimedean local field, e.g. the p-adic numbers $\mathbb{Q}_p$, a notion that we will introduce in Section 2.2 and that plays a central role in number theory. The building blocks for these representations are called *supercuspidal representations* (defined in Section 2.4) and until not too long ago surprisingly little was known about these representations for general p-adic groups. A construction of supercuspidal representations of p-adic groups lays not only the foundation for work within the representation theory of p-adic groups but also allows for a plethora of applications far beyond this area, for example to advance different incarnations of the Langlands program, including the local, global and relative Langlands program.

In 1977, a Symposium in Pure Mathematics was held in Corvallis that led to famous Proceedings. One of the articles in the Proceedings was entitled "Representations of p-adic groups: A survey", written by Cartier ([Car79b]). We quote from the introduction of this article:

> "The main goal of this article will be the description and study of the principal series and the spherical functions. There shall be almost no mention of two important lines of research which are still actively pursued today:
> (a) [...]
> (b) *Explicit construction of absolutely cuspidal representations [nowadays usually called "supercuspidal representations"].* Here important progress has been made by Shintani [Shi68], Gérardin [Gér75] and Howe (forthcoming papers in the Pacific J. Math.). One can expect to meet here difficult and deep arithmetical questions which are barely uncovered."

The present survey will focus on the developments of an explicit construction of supercuspidal representations. It will provide an introduction to the groundbreaking methods introduced since the above Symposium had happened to tackle the construction of supercuspidal representations for general p-adic groups and conclude with new developments of the last five years. Thereby the present survey complements the above survey by Cartier, which focuses on how to reduce the classification of representations of p-adic groups to the back then unknown construction of supercuspidal representations. (We will not assume that the reader has read the above survey by Cartier.)

Since the work mentioned in the above quote that started about 50 years ago, mathematicians have tried to construct these mysterious supercuspidal representations. To mention a few, in 1979, Carayol ([Car79a]) gave a construction of all supercuspidal representations of the general linear group $\mathrm{GL}_n(F)$ for n a prime number different from p, the residue field characteristic of F (i.e. the "p" in "p-adic"). In 1986, Moy ([Moy86a]) proved that Howe's construction ([How77]) from the 1970s exhausts all supercuspidal representations of $\mathrm{GL}_n(F)$ if n is coprime to p. In the early 1990s, Bushnell and Kutzko extended these constructions to obtain all supercuspidal representations of $\mathrm{GL}_n(F)$ for arbitrary n ([BK93]). These results play a crucial role in the representation theory of $\mathrm{GL}_n(F)$. Similar methods have been exploited by Stevens ([Ste08]) around 15 years ago to construct all supercuspidal representations of classical groups for $p \neq 2$, i.e. orthogonal, symplectic and unitary groups. His work was preceded by a series of partial results by Moy ([Moy86b] for $\mathrm{U}(2,1)$, [Moy88] for GSp_4), Morris ([Mor91] and [Mor92]) and Kim ([Kim99]). Moreover, Zink ([Zin92]) treated division algebras over non-archimedean local fields of characteristic zero, Broussous ([Bro98]) treated division algebras without restriction on the characteristic, and Sécherre and Stevens ([SS08]) completed the case of all inner forms of $\mathrm{GL}_n(F)$ about 15 years ago. The construction of supercuspidal representations for inner forms of $\mathrm{GL}_n(F)$ plays a key role in the explicit description of the local Jacquet–Langlands correspondence, which is an instance of Langlands functoriality.

For arbitrary reductive groups the story is less complete. The introduction of the Moy–Prasad filtration in the 1990s spurred remarkable progress. The work of Moy and Prasad built upon the innovative Bruhat–Tits theory introduced in the 1970s/1980s: In [BT72, BT84], Bruhat and Tits defined a building $\mathscr{B}(G,F)$ associated to a p-adic group $G(F)$ on which the p-adic group $G(F)$ acts. For each point x in $\mathscr{B}(G,F)$, they constructed a compact subgroup $G_{x,0}$ of $G(F)$, called a *parahoric subgroup*, which is (up to finite index and center) the stabilizer G_x of the point x in G. In [MP94, MP96], Moy and Prasad defined a filtration of these parahoric subgroups by smaller normal subgroups

$$G_{x,0} \triangleright G_{x,s_1} \triangleright G_{x,s_2} \triangleright G_{x,s_3} \triangleright \ldots,$$

where $0 < s_1 < s_2 < \ldots$ are real numbers depending on x. These subgroups play a crucial role in the study and construction of supercuspidal representations and will be introduced in Section 3 below.

For example, if we take $G = \mathrm{SL}_2$ over the field $F = \mathbb{Q}_p$, the p-adic numbers, with ring of integers $\mathbb{Z}_p$, then the Bruhat–Tits building is an infinite tree with valency $p+1$, see Figure 1 (for $p = 2$). Let y be the barycenter of a maximal facet, i.e. the center of an edge of the infinite tree, and x a vertex of the the edge. Then (by choosing an appropriate basis) the associated Moy–Prasad filtrations at the points x and y look like the following (where we

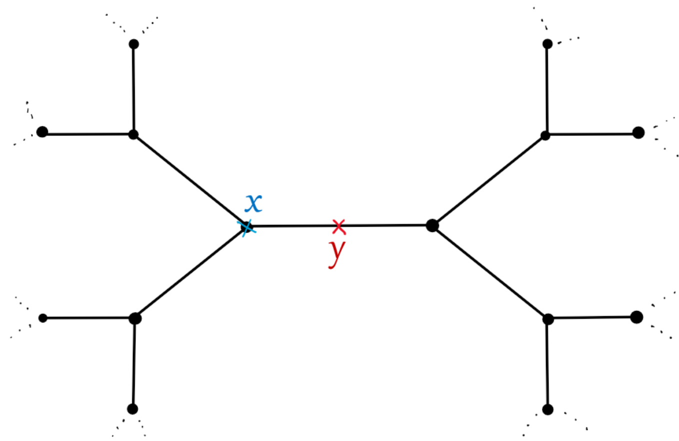

Figure 1: Bruhat–Tits building for $\mathrm{SL}_2(\mathbb{Q}_2)$.

intersect the displayed matrices with $\mathrm{SL}_2(\mathbb{Q}_p)$):

$$G_{x,0} = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \qquad G_{y,0} = \begin{pmatrix} \mathbb{Z}_p & p\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$$

$$G_{y,0.5} = \begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ \mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}$$

$$G_{x,1} = \begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix} \qquad G_{y,1} = \begin{pmatrix} 1+p\mathbb{Z}_p & p^2\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}$$

$$G_{y,1.5} = \begin{pmatrix} 1+p^2\mathbb{Z}_p & p^2\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p^2\mathbb{Z}_p \end{pmatrix}$$

$$G_{x,2} = \begin{pmatrix} 1+p^2\mathbb{Z}_p & p^2\mathbb{Z}_p \\ p^2\mathbb{Z}_p & 1+p^2\mathbb{Z}_p \end{pmatrix} \qquad G_{y,2} = \begin{pmatrix} 1+p^2\mathbb{Z}_p & p^3\mathbb{Z}_p \\ p^2\mathbb{Z}_p & 1+p^2\mathbb{Z}_p \end{pmatrix}$$

$$\vdots \qquad\qquad\qquad \vdots$$

Based on this filtration, Moy and Prasad introduced in [MP94, MP96] the notion of *depth* of a representation, which measures the first occurrence of a fixed vector in a given representation. The precise definition will be introduced in Section 3.4. In [MP96], Moy and Prasad gave a classification of depth-zero representations, showing, roughly speaking, that they correspond to representations of finite groups of Lie type, the group $G_{y,0}/G_{y,0+}$. A similar result was obtained around the same time by Morris ([Mor99]). We will discuss depth-zero representations in more detail in Section 4.4.

In 1998, Adler used the Moy–Prasad filtration to provide a construction of positive-depth supercuspidal representations for general p-adic groups (that split over a tamely ramified extension), which was generalized by Yu ([Yu01]) in 2001. Since then, Yu's construction has been widely used, e.g. to study the Howe correspondence ([LM18]), to understand distinction of representations of p-adic groups, i.e. the question if the restriction of a representation to a subgroup contains the trivial representation ([HM08, HL12, Hak13, Zha15, Zha20]), to obtain character formulas ([AS09, DS16, Spi18, Spi, FKS]) and to construct an explicit local Langlands correspondence ([Kal19, Kal]). We will sketch Yu's construction in Section 4.

Given the importance of having an explicit construction of supercuspidal representations, Kim ([Kim07]) achieved the subsequent breakthrough in 2007 by proving that if F has characteristic zero and the prime number p is "very large", then all supercuspidal representations arise from Yu's construction. Last year, in 2021, it has been shown ([Fin21d]) via very different techniques that Yu's construction provides us with all supercuspidal representations only under the minor assumption that p does not divide the order of the (absolute) Weyl group of the (tame) p-adic group, an invariant attached to the p-adic group that we will introduce in Section 2.1. In particular, the result also works for fields F of positive characteristic. We will provide some more details in Section 5. Based on [Fin21c], we expect this result to be

type	$A_n\,(n \geq 1)$	$B_n, C_n\,(n \geq 2)$	$D_n\,(n \geq 3)$	E_6	E_7
order	$(n+1)!$	$2^n \cdot n!$	$2^{n-1} \cdot n!$	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$

type	E_8	F_4	G_2
order	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^7 \cdot 3^2$	$2^2 \cdot 3$

Table 1: Order of irreducible Weyl groups ([Bou02, VI.4.5-VI.4.13])

essentially optimal (when considering also types for non-supercuspidal Bernstein blocks and treating all inner forms together, the details of which we omit from this survey).

In fact, in 2014, Reeder and Yu ([RY14]) gave a new construction of some supercuspidal epipelagic representations of tame semisimple groups, which generalizes the simple supercuspidal representations previously constructed by Gross and Reeder ([GR10]). Epipelagic representations are representations of smallest positive depth. The papers of Fintzen–Romano ([FR17], special case) and Fintzen ([Fin21b], general case) show that the input for Reeder and Yu's construction also exists for small primes p, which provided examples of positive-depth supercuspidal representations that do not arise from Yu's construction. It is current work in progress to provide a more general construction that also works for small p.

In this survey, we will focus on the known construction of supercuspidal representations under the assumption that p does not divide the order of the Weyl group. While Yu ([Yu01, Fin21a]) showed how to construct a supercuspidal representation from a given input (spelled out in Section 4.1), Hakim and Murnaghan ([HM08] answered the questions of which inputs yield the same supercuspidal representations (see Section 4.5 for the answer), which thus leads to a parametrization of supercuspidal representations. However, it was recently suggested by Fintzen, Kaletha and Spice ([FKS]) to twist Yu's construction by a quadratic character, i.e. a character of an appropriate compact open subgroup appearing in the construction that takes values in $\{\pm 1\}$. While at first glance this just looks like changing the parametrization of supercuspidal representations, the existence of the quadratic character has far-reaching consequences. For example, it allowed to calculate formulas for the Harish-Chandra character of these supercuspidal representations ([FKS, Spi]), to write down a candidate for the local

Langlands correspondence for simple supercuspidal representations ([Kal]) and to prove that the local Langlands correspondence for regular supercuspidal representations introduced by Kaletha ([Kal19]) satisfies the desired character identities ([FKS]).

2 What are p-adic groups and representations of p-adic groups?

This section will give an introduction to p-adic groups. Those who understand the following sentence may skip this section: A p-adic group is the group of F-points of a connected reductive group over a non-archimedean local field F. Those who see the notion of a reductive group for the first time are encouraged to pay particular attention to the examples we introduce below. Reading the remainder of the survey focusing on a few examples rather than the general notion should allow the reader to get some feel for the topic. We also warn the reader that we have not chosen the most general and most modern treatment, but instead an approach that requires less prerequisites and space. Some of the definitions we make are often not used as initial definitions in text books but rather stated as being an equivalent characterization in later theorems.

2.1 Reductive groups over algebraically closed fields

Let k be an algebraically closed field. In this section we give an overview of the notion of a reductive group and its important structural properties. For more details, see, for example, the classical text books on Linear Algebraic Groups [Bor91, Hum75, Spr09] or Brian Conrad's lecture notes ([Con17a, Con17b]) for a more modern treatment.

Definition 2.1.1. A *linear algebraic group* over k is a reduced Zariski-closed subgroup of the general linear group GL_n over k for some integer n. (Equivalently, a *linear algebraic group* is a smooth affine group scheme over k.)

Let G be (the k-points of a) linear algebraic group over k. To simplify notation, in this section we will not distinguish between the linear algebraic groups and their k-points, but still secretly remember the variety structure when talking about k-points. When we talk about subgroups in this section, we always mean reduced closed subvarieties that are also subgroups (in other words closed subgroupschemes endowed with the reduced subscheme structure) unless explicitly stated otherwise.

Let us begin with a list of examples of linear algebraic groups to keep in mind throughout the survey:

- the general linear group $\mathrm{GL}_n(k)$, i.e. $n \times n$ invertible matrices with entries in k
- the special linear group $\mathrm{SL}_n(k)$, i.e. matrices of determinant one

- the subgroup $N_n(k)$ of $\mathrm{GL}_n(k)$ consisting of matrices of the following shape

$$\begin{pmatrix} 1 & \star & \star & \dots & \star \\ 0 & 1 & \star & \dots & \star \\ 0 & \ddots & \ddots & \ddots & \star \\ 0 & \dots & 0 & 1 & \star \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

- the subgroup $B_n(k)$ of $\mathrm{GL}_n(k)$ consisting of matrices of the following shape

$$\begin{pmatrix} \star & \star & \star & \dots & \star \\ 0 & \star & \star & \dots & \star \\ 0 & \ddots & \ddots & \ddots & \star \\ 0 & \dots & 0 & \star & \star \\ 0 & \dots & 0 & 0 & \star \end{pmatrix}$$

- the orthogonal group $\mathrm{O}_n(k) = \{A \in \mathrm{GL}_n(k) \mid {}^tAA = 1\}$, where ${}^tA$ denotes the transpose of A and 1 denotes the identity matrix consisting of ones on the diagonal and zeros everywhere else

- the special orthogonal group $\mathrm{SO}_n(k) = \{A \in \mathrm{O}_n(k) \mid \det(A) = 1\}$

- the symplectic group $\mathrm{Sp}_{2n}(k) = \{A \in \mathrm{GL}_n(k) \mid {}^tAJA = J\}$, where $J = \begin{pmatrix} 0_{n\times n} & w_n \\ w_n & 0_{n\times n} \end{pmatrix}$ with $w_n = \begin{pmatrix} & & & 1 \\ & & \iddots & \\ & 1 & & \\ 1 & & & \end{pmatrix}$

In representation theory we often restrict to a subclass of linear algebraic groups, called reductive groups. In order to define them, we need to introduce the notion of unipotent groups and unipotent radicals.

Definition 2.1.2. A (closed reduced) subgroup G of $\mathrm{GL}_n(k)$ is called *unipotent* if G is conjugate to a subgroup of $N_n(k) = \left\{ \begin{pmatrix} 1 & \star & \star & \dots & \star \\ 0 & 1 & \star & \dots & \star \\ 0 & \ddots & \ddots & \ddots & \star \\ 0 & \dots & 0 & 1 & \star \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_n(k)$

Definition 2.1.3. The *unipotent radical* $(RG)_u$ (or R_uG) of a linear algebraic group G is the maximal connected unipotent normal subgroup of G.

Definition 2.1.4. A linear algebraic group G is *reductive* if its unipotent radical is trivial.

Examples of reductive groups include $\mathrm{GL}_n(k), \mathrm{SL}_n(k), \mathrm{O}_n(k), \mathrm{SO}_n(k), \mathrm{Sp}_{2n}(k)$ and products of reductive groups. The group GL_1 is often also called the multiplicative group, because $\mathrm{GL}_1(k) = k^\times$ (with group law multiplication), and we also denote it by $\mathbb{G}_m$.

Examples of linear algebraic groups that are not reductive include the groups $N_n(k)$ and $B_n(k)$ for $n \geq 2$. In both cases their unipotent radical is $N_n(k)$, which is nontrivial. The group N_2 is also called the additive group, since $N_2(k) = k$ (with group law addition), and we also denote it by $\mathbb{G}_a$.

Reductive groups have a rather rich structure, similarly to compact Lie groups, which forms a basis for studying the representation theory of these groups. A key tool to obtain this structure is to consider the following objects.

Definition 2.1.5. A *torus* is a product of multiplicative groups, i.e. $\mathbb{G}_m \times \ldots \times \mathbb{G}_m = (\mathbb{G}_m)^n$ for some integer n. We say that a subgroup T of a linear algebraic group G is a *maximal torus* if T is a torus and is not strictly contained in a larger torus that is also a subgroup of G.

A crucial theorem for the structure theory is the uniqueness of maximal tori up to conjugation.

Theorem 2.1.6. *All maximal tori in G are conjugate.*

For example, all maximal tori of $\mathrm{GL}_n(k)$ are conjugate to the group of diagonal matrices

$$\left\{ \begin{pmatrix} \star & & \\ & \ddots & \\ & & \star \end{pmatrix} \right\}.$$

Since linear algebraic groups are varieties we also have the powerful tool of considering the tangent space of the variety at points of our choice. The canonical point to choose is the identity element of the group $G(k)$. Since linear algebraic groups are not just varieties, but also groups, the tangent space $T(G)_e$ of the group G at the identity e can be equipped with the structure of a Lie algebra. We denote the resulting Lie algebra by $\mathrm{Lie}(G)(k)$. The Lie bracket for $\mathrm{Lie}(\mathrm{GL}_n)(k) = \mathrm{Mat}_{n\times n}(k)$ is given by $[A, B] = AB - BA$ for $A, B \in \mathrm{Mat}_{n\times n}(k)$, which restricts to the Lie bracket for $\mathrm{Lie}(G)(k)$ when G is a subgroup of GL_n. We refer the reader to the literature for the general abstract definition and to Table 2 for some examples. We will often denote the Lie algebra $\mathrm{Lie}(G)(k)$ of $G(k)$ also by $\mathfrak{g}(k)$ and in general use fraktur letters denote the Lie algebra, i.e. $\mathfrak{gl}_n(k)$ will denote the Lie algebra of $\mathrm{GL}_n(k)$, etc.

The group $G(k)$ acts on itself by conjugation. Taking the derivative of this action, we obtain an action of $G(k)$ on its Lie algebra $\mathfrak{g}(k)$, which we call the *adjoint action* and denote by Ad. For example, for $g \in \mathrm{GL}_n(k)$ and $A \in \mathrm{Lie}(\mathrm{GL}_n)(k) = \mathrm{Mat}_{n\times n}(k)$, we have

$$\mathrm{Ad}(g)(A) = gAg^{-1}.$$

$G(k)$	$\mathrm{Lie}(G)(k)$	Lie bracket $[\cdot,\cdot]$
$\mathrm{GL}_n(k)$	$\mathrm{Mat}_{n\times n}(k)$	$[A,B]=AB-BA$
$\mathrm{SL}_n(k)$	$\mathrm{Mat}_{n\times n}(k)_{\text{trace}=0}$	$[A,B]=AB-BA$
$\mathrm{Sp}_{2n}(k)$	$\{A\in\mathrm{Mat}_{n\times n}(k)\mid JA+{}^tAJ=0\}$	$[A,B]=AB-BA$

Table 2: Examples of Lie algebras

Let T be a maximal torus of G. Then $T(k)$ acts on $\mathfrak{g}(k)$ via the (restriction to $T(k)$ of the) adjoint action. This action is diagonalizable, i.e. we can decompose $\mathfrak{g}(k)$ into a sum of simultaneous eigenspaces for the action of $T(k)$. The "eigenvalues" in this setting are then characters of the torus.

For example, if $G(k)=\mathrm{GL}_n(k)$, we may choose T to be the subgroup consisting of diagonal matrices, i.e.

$$T(k)=\left\{\begin{pmatrix}t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n\end{pmatrix}\mid t_1,t_2,\dots t_n\in k^\times\right\},$$

and let $X_{i,j}$ for $1\le i,j\le n$ be the matrix with a one in position (i,j) and zeros everywhere else. Then

$$\mathrm{Ad}(\mathrm{diag}(t_1,t_2,\dots,t_n))(X_{i,j})=t_it_j^{-1}X_{i,j}$$

and $\mathfrak{gl}_n(k)=\oplus_{1\le i,j\le n}kX_{i,j}$.

Notation 2.1.7. We write $X^*(T)=\mathrm{Hom}_k(T,\mathbb{G}_m)$ for the homomorphisms from T to $\mathbb{G}_m$ as group varieties i.e. morphisms of algebraic varieties that commute with the group action. The group law on $\mathbb{G}_m$ turns $X^*(T)$ into an abelian group.

Note that if $T\simeq\mathbb{G}_m$, then $X^*(\mathbb{G}_m)\simeq\mathrm{Hom}_k(\mathbb{G}_m,\mathbb{G}_m)\simeq\mathbb{Z}$ (with the group law on $\mathbb{Z}$ being addition), where the isomorphism is given by sending an integer $n\in\mathbb{Z}$ to the element $f_n\in\mathrm{Hom}_k(\mathbb{G}_m,\mathbb{G}_m)$ that satisfies $f_n(x)=x^n$ for $x\in\mathbb{G}_m(k)=k^\times$. More generally, we have by definition that $T\simeq\mathbb{G}_m^n$ for some integer n, and hence

$$X^*(T)\simeq\mathrm{Hom}_k(\mathbb{G}_m^n,\mathbb{G}_m)\simeq\mathbb{Z}^n.$$

From now on we assume that G is a reductive group, and we write $\mathfrak{g}(k)=\oplus_{\alpha\in X^*(T)}\mathfrak{g}_\alpha(k)$, where

$$\mathfrak{g}_\alpha(k)=\{X\in\mathfrak{g}(k)\mid \mathrm{Ad}(t)(X)=\alpha(t)X\ \forall t\in T(k)\}.$$

Definition 2.1.8. The *roots* of G with respect to T are the elements

$$\Phi(G,T)=\{\alpha\in X^*(T)\setminus\{0\}\mid \dim\mathfrak{g}_\alpha(k)\neq 0\}\subset X^*(T)\subset X^*(T)\otimes_{\mathbb{Z}}\mathbb{R}\simeq\mathbb{R}^n$$

for some integer n.

Fact 2.1.9. *(a) The subspace $\mathfrak{g}_\alpha(k)$ for $\alpha \in \Phi(G,T)$ is one dimensional.*

(b) The subset $\Phi(G,T)$ of the real vector space $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ forms a root system, and this root system does not depend on the choice of T.

While we will not use this fact later, we remark that $G/Z(G)$, where $Z(G)$ denotes the scheme-theoretic center (a not necessarily reduced/smooth subgroup), is uniquely determined by the root system.

Examples.

$\mathrm{GL}_n(k)$**:** Let $G(k) = \mathrm{GL}_n(k)$ with T the torus above consisting of diagonal matrices. Then $\Phi(G,T) = \{\alpha_{i,j} \,|\, 1 \le i, j \le n, i \ne j\}$, where $\alpha_{i,j}$ is the character of T that satisfies

$$\alpha_{i,j} : \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \mapsto t_i t_j^{-1}.$$

Note that all roots of $\mathrm{GL}_3(k)$ lie in a plane. Restricting our attention to this plane the root system is drawn in Figure 2.

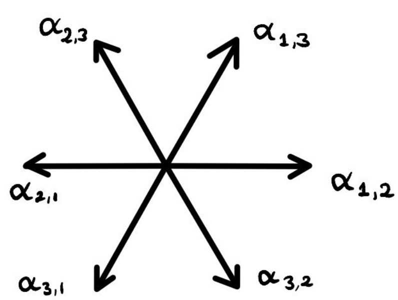

Figure 2: Root system of GL_3

$\mathrm{Sp}_{2n}(k)$**:** Let $G(k) = \mathrm{Sp}_{2n}(k)$ with T the subgroup consisting of diagonal matrices, i.e.

$$T(k) = \left\{ \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} \,\middle|\, t_1, t_2, \dots t_n \in k^\times \right\},$$

Then $\Phi(G,T) = \{\alpha_{\pm i, \pm j}, \alpha_{\pm i}, \alpha_{\pm n} \,|\, 1 \le i < j \le n, i \ne j\}$, where $\alpha_{\pm i, \pm j}$ is the character of T that satisfies

$$\alpha_{\pm i, \pm j} : \mathrm{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) \mapsto t_i^{\pm 1} t_j^{\pm 1}$$

and $\alpha_{\pm i}$ is the character satisfying

$$\alpha_{\pm i} : \mathrm{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \mapsto t_i^{\pm 2}.$$

Figure 3 shows the root system of Sp_4.

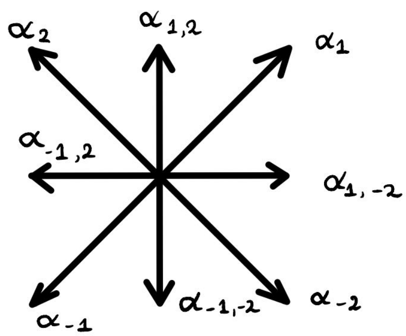

Figure 3: Root system of Sp_4

Definition 2.1.10. We call a subset Δ of $\Phi(G,T)$ a *basis* of $\Phi(G,T)$ if every root $\alpha \in \Phi(G,T)$ can be written uniquely as a sum $\sum_{\alpha_i \in \Delta} n_i \alpha_i$ where either all the n_i are non-negative integers or all the n_i are non-positive integers.

For example, if $G(k) = \mathrm{GL}_n(k)$ with T and $\Phi(G,T)$ as above, then we can choose

$$\Delta = \{\alpha_{i,i+1} \,|\, 1 \leq i \leq n-1\}.$$

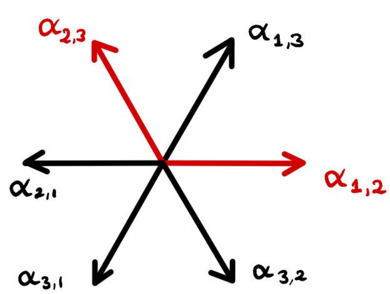

Figure 4: Root system of GL_3 with a choice of basis $\Delta = \{\alpha_{1,2}, \alpha_{2,3}\}$ in red

In order to classify and construct representations of reductive groups the following subgroups will become important.

Definition 2.1.11. A *Borel subgroup* is a maximal connected solvable (closed reduced) subgroup of G. A *parabolic* subgroup is a (closed reduced) subgroup of G that contains a Borel subgroup.

Fact 2.1.12. *All Borel subgroups of a reductive group are conjugate.*

For $\mathrm{GL}_n(k)$, the Borel subgroups are the conjugates of

$$B_n(k) = \left\{ \begin{pmatrix} \star & \star & \star & \dots & \star \\ 0 & \star & \star & \dots & \star \\ 0 & \ddots & \ddots & \ddots & \star \\ 0 & \dots & 0 & \star & \star \\ 0 & \dots & 0 & 0 & \star \end{pmatrix} \right\}$$

and the parabolic subgroups are conjugates of block upper triangular matrices, e.g. matrices of the shape

$$\left\{ \begin{pmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & \star \end{pmatrix} \right\} \text{ or } \left\{ \begin{pmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \end{pmatrix} \right\}.$$

The parabolic subgroups of a reductive group G that contain a fixed Borel subgroup $B \subset G$ are in one to one correspondence with subsets of a basis Δ of $\Phi(G, T)$.

Fact 2.1.13. *Let P be a parabolic subgroup of G.*

(i) P is reductive if and only if $P = G$.

(ii) There exists a Levi decomposition of the parabolic subgroup P, i.e. we can write P as a semidirect product $M \ltimes N$, where M is a reductive group and N is the unipotent radical of P. M is called a Levi subgroup of P. Note that M is not unique in general.

For example, the Levi decomposition $P = M \ltimes N$ for a parabolic subgroup of $\mathrm{GL}_n(k)$ might look like the following

$$\left\{ \begin{pmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & \star \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \star & \star & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 & 0 \\ 0 & 0 & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star & 0 \\ 0 & 0 & 0 & 0 & 0 & \star \end{pmatrix} \right\} \ltimes \left\{ \begin{pmatrix} 1 & 0 & \star & \star & \star & \star \\ 0 & 1 & \star & \star & \star & \star \\ 0 & 0 & 1 & 0 & 0 & \star \\ 0 & 0 & 0 & 1 & 0 & \star \\ 0 & 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

In order to construct representations, we need to get an even better handle on the structure of reductive groups. Apart from the tori, the below defined root groups will play a crucial role. We fix a maximal torus T of our reductive group G and recall that $\mathfrak{g}_\alpha(k) = \{X \in \mathfrak{g}(k) \mid \mathrm{Ad}(t)(X) = \alpha(t)X \ \forall t \in T(k)\}$

Definition (Fact) 2.1.14. Let $\alpha \in \Phi(G,T)$. The *root (sub)group* U_α is the unique (closed reduced) connected T-stable subgroup of G whose Lie algebra is $\mathfrak{g}_\alpha$.

The root group U_α is isomorphic to the additive group $\mathbb{G}_a$. For $G = \mathrm{GL}_n$, the root subgroup $U_{\alpha_{i,j}}(k)$ consists of those matrices that have ones on the diagonal and zeros in all non-diagonal entries except for the (i,j)-entry. For example, for $G = \mathrm{GL}_2$,

$$U_{\alpha_{1,2}}(k) = \left\{ \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \right\}.$$

Fact 2.1.15. *Let B be a Borel subgroup containing a maximal torus T of G. Then $B = T \ltimes U$ and U is isomorphic as a variety (not necessarily as a group) to the product variety $\prod_{\alpha \in \Phi^+} U_\alpha$, where Φ^+ denotes all those roots that are a non-negative linear combination of roots in an appropriately chosen basis $\Delta \subset \Phi(G,T)$.*

Definition 2.1.16. We write $N(T)$ for the subgroup of G that normalizes the torus maximal torus T. The *Weyl group* W is defined to be the quotient $N(T)/T$ of the normalizer of T by the torus T.

The Weyl group is a finite group. More precisely, we have the following fact.

Fact 2.1.17. *The group $W = N(T)/T$ is the Weyl group of the root system $\Phi(G,T)$, i.e. it is the subgroup of the isometries of the real vector space $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by the reflections $\{s_\alpha \mid \alpha \in \Delta\}$, where s_α denotes the reflection about the hyperplane perpendicular to the root α and Δ is a basis for $\Phi(G,T)$.*

If $G = \mathrm{GL}_n$, the Weyl group is isomorphic to the symmetric group S_n on a set of n elements. If T is the torus consisting of diagonal matrices, then representatives for the elements in the Weyl group can be chosen to be permutation matrices, e.g. for GL_2 the nontrivial element in the Weyl group can be represented by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

It is a nice exercise to observe that the Weyl group of the symplectic group Sp_{2n} is isomorphic to the semidirect product $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$.

We conclude this section by stating an important decomposition of reductive groups into locally closed subsets.

Theorem 2.1.18 (Bruhat decomposition)**.** *The group G is (as a set) the disjoint union $\sqcup_{w \in W} B n_w B$, where $n_w \in N(T)$ is any element whose imagine in $W = N(T)/T$ is w and B denotes a Borel subgroup of G containing T. Moreover, multiplication yields an isomorphism of varieties from*

$$B \times n_w \times \prod_{\substack{\alpha \in \Phi^+ \\ w^{-1}(\alpha) \notin \Phi^+}} U_\alpha \xrightarrow{\simeq} B n_w B.$$

2.2 p-adic numbers and other non-archimedean local fields

Definition 2.2.1. The *p-adic absolute value* of a rational number $p^s \cdot \frac{a}{b}$ with a and b non-zero integers coprime to p and s an arbitrary integer is defined by

$$\left|p^s \cdot \frac{a}{b}\right|_p = \left(\frac{1}{p}\right)^s$$

and $|0| = 0$.

The p-adic absolute value on the rational numbers is a *non-archimedean absolute value*, i.e. $|x + y|_p \leq \max(|x|_p, |y|_p)$ for all rational numbers x and y.

Definition 2.2.2. The *p-adic integers* $\mathbb{Z}_p$ are the completion of the integers $\mathbb{Z}$ by the p-adic absolute value $|\cdot|_p$.

The *p-adic numbers* $\mathbb{Q}_p$ are the completion of the rational numbers $\mathbb{Q}$ by the p-adic absolute value $|\cdot|_p$.

This means we can represent p-adic integers as a converging "power series in p":

$$a_0 + a_1 \cdot p + a_2 \cdot p^2 + a_3 \cdot p^3 + \ldots \text{ for some integers } a_i \ (0 \leq a_i < p),$$

and we can write a p-adic number as a "Laurent series in p" (with only finitely many terms with negative exponents):

$$a_{-n} \cdot p^{-n} + \ldots + a_0 + a_1 \cdot p + a_2 \cdot p^2 + \ldots \mid a_i \ (0 \leq a_i < p).$$

A ring closely related to the ring of p-adic integers is the ring of power series $\mathbb{F}_p[[t]]$ with coefficients in a finite field $\mathbb{F}_p$ with p elements. We denote its fraction field by $\mathbb{F}_p((t))$. It is the field of *formal Laurent series over* $\mathbb{F}_p$, i.e. its elements can be written as

$$a_{-n} \cdot t^{-n} + \ldots + a_0 + a_1 \cdot t + a_2 \cdot t^2 + \ldots \mid a_i \in \mathbb{F}_p,$$

where we only allow finitely many non-zero coefficients for the negative exponents of t. We equip the field $\mathbb{F}_p((t))$ with the absolute value satisfying

$$\left|t^s(a_0 + a_1 \cdot t + a_2 \cdot t^2 + \ldots)\right|_p = \left(\frac{1}{p}\right)^s$$

for $a_0 \in \mathbb{F}_p \setminus \{0\}$, $a_1, a_2, \ldots \in \mathbb{F}_p$ and any integer s.

Let E be a finite field extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Then we can extend the absolute value $|\cdot|_p$ (uniquely) to an absolute value on E, which we also denote by $|\cdot|_p$. This absolute value allows us to equip E with a topology.

Definition 2.2.3. A *non-archimedean local field* is a finite field extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ equipped with the topology arising from the absolute value $|\cdot|_p$.

Notation 2.2.4. The *ring of integers* $\mathcal{O}_E$ of E is the subring:

$$\mathcal{O}_E = \{e \in E \mid |e|_p \leq 1\}$$

and it has the maximal ideal

$$\mathcal{P}_E = \{e \in E \mid |e|_p < 1\}.$$

An element $\varpi_E \in \mathcal{P}_E$ whose p-adic absolute value is maximal among the elements in $\mathcal{P}_E$ is called a *uniformizer.*

2.3 p-adic groups

In this section we define reductive groups over non-algebraically closed fields. Let F be either a non-archimedean local field or a finite field, and fix an algebraic closure $\overline{F}$ of F. We will view all algebraic field extensions of F as contained in $\overline{F}$.

We first state the definition of a linear algebraic group over F and then provide some explanation for those who have not seen the notion of geometrically reduced closed F-subgroups before.

Definition 2.3.1. A *linear algebraic group* over F is a geometrically reduced closed F-subgroup of the general linear group GL_n over F for some integer n. (Equivalently, a *linear algebraic group* is a smooth affine group scheme over F.)

To explain what we mean by an F-subgroup, we note that the ring of regular functions of GL_n over $\overline{F}$ is given by

$$\overline{F}[\mathrm{GL}_n] := \overline{F}[x_{i,j}, y \,|\, 1 \leq i, j \leq n]/(\det(x_{i,j})y - 1),$$

which can be written as

$$\overline{F}[\mathrm{GL}_n] = F[\mathrm{GL}_n] \otimes_F \overline{F}, \text{ where } F[\mathrm{GL}_n] := F[x_{i,j}, y \,|\, 1 \leq i, j \leq n]/(\det(x_{i,j})y - 1).$$

Definition 2.3.2. An ideal $I \subset \overline{F}[\mathrm{GL}_n]$ is *defined over* F if $I \cap F[\mathrm{GL}_n]$ generates I as an ideal.

This allows us to restate the definition of a linear algebraic group over F.

Definition 2.3.3. A *linear algebraic group* over F is a reduced closed subgroup of GL_n over $\overline{F}$ that is defined as the set of zeros of some ideal $I \subset \overline{F}[\mathrm{GL}_n]$ that is defined over F.

Let G be a linear algebraic group over F. For an algebraic field extension F'/F, we write $G(F')$ for the F'-points of G, i.e. the intersection $G(\overline{F}) \cap \mathrm{GL}_n(F')$. We denote by $G_{F'}$ the base change of G to F', which means that we only remember that the group is defined over F' rather than F.

Definition 2.3.4. A linear algebraic group G over F is *reductive* if $G_{\overline{F}}$ is reductive, i.e. the unipotent radical $(RG_{\overline{F}})_u$ of $G_{\overline{F}}$ is trivial.

This allows us to now understand what we mean by a p-adic group. A *p-adic group* is the group of F-points of a connected reductive group over a non-archimedean local field F. We caution the reader that different authors might mean different things by "p-adic groups", e.g. some only work with reductive groups over finite extensions of the p-adic numbers.

Examples of reductive groups over F include the groups $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{O}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n}$ that can all be defined by ideals over F. However, new interesting phenoma arise over non-algebraic closed fields.

Definition 2.3.5. An *F-torus* (or a torus over F) is a linear algebraic group T over F such that $T_{\overline{F}}$ is a torus, i.e. such that $T_{\overline{F}}^n \simeq \mathbb{G}_m^n$ for some integer n.

As over algebraically closed fields, we denote GL_1 over F also by $\mathbb{G}_m$. Then for every field extension F'/F, we have $\mathbb{G}_m(F') = (F')^\times$. Taking products of the multiplicative group $\mathbb{G}_m$ provides us with examples of tori, which we call *split tori*.

Definition 2.3.6. An F-torus T is called *split (or F-split)* if $T \simeq \mathbb{G}_m^n$ for some integer n.

All tori over algebraically closed fields are split, however, over non-algebraically closed fields, the theory is richer and becomes a key ingredient for the construction of representations of p-adic groups. Here is an example of a torus that is used in the construction of supercuspidal representations.

Example of a non-split torus. Let $F = \mathbb{Q}_p$, and let E be the quadratic extension of $\mathbb{Q}_p$ obtained by adjoining a squareroot $\sqrt{p}$ of p. We define T to be the torus over F that is a subgroup of GL_2 satisfying

$$T(\overline{F}) = \left\{ \begin{pmatrix} a & b \\ pb & a \end{pmatrix} \subset \mathrm{GL}_2(\overline{F}) \right\}.$$

Then

$$T(F) = \left\{ \begin{pmatrix} a & b \\ pb & a \end{pmatrix} \subset \mathrm{GL}_2(F) \right\} \simeq E^\times \not\simeq (F^\times)^2.$$

An important result by Grothendieck that allows us to understand the structure of reductive groups over F is the following.

Theorem 2.3.7. *If G is a linear algebraic group, then there exists an F-torus $T \subset G$ such that $T_{\overline{F}}$ is a maximal torus of $G_{\overline{F}}$.*

Definition 2.3.8. A reductive group G over F is called *split* (or *F-split*) if it contains a maximal torus that is split.

For split reductive groups we obtain an analogous structure theory to the one discussed above over the algebraic closure. In particular, Borel subgroups are defined over F, the Lie algebra of the reductive group decomposes into the Lie algebra of a torus and one-dimensional subalgebras $\mathfrak{g}_\alpha(F)$ on which the torus acts via a root α, and root groups U_α are defined over F and isomorphic to the additive group $\mathbb{G}_a$ over F.

2.4 Representations of p-adic groups

From now on let F be a non-archimedean local field and let G be a connected reductive group over F. We equip $G(F)$ with the topology arising from the topology of F, i.e. a basis of open neighborhoods of the identity 1 in $\mathrm{GL}_n(F)$ consists of the subgroups

$$1 + \varpi \operatorname{Mat}_{n\times n}(\mathcal{O}_F) \supset 1 + \varpi^2 \operatorname{Mat}_{n\times n}(\mathcal{O}_F) \supset 1 + \varpi^3 \operatorname{Mat}_{n\times n}(\mathcal{O}_F) \supset \dots,$$

where ϖ denotes a uniformizer of F. Then $G(F)$ is the group that we also refer to as a p-adic group.

Definition 2.4.1. A *smooth representation* (π, V) of $G(F)$ is

- a complex vector space V and
- a group homomorphism $\pi : G(F) \to \mathrm{Aut}(V)$

such that for every $v \in V$ the stabilizer $\mathrm{Stab}(v) = \{g \in G(F) \,|\, \pi(g)v = v\}$ of v in $G(F)$ is an open subset of $G(F)$.

We define smooth representations of closed subgroups of $G(F)$ (with respect to the p-adic topology underlying the topological group $G(F)$) analogously.

In this survey we will focus on the *irreducible* smooth representations, i.e. those smooth representations (π, V) that have precisely two subrepresentations (subspaces of V preserved under the action of $G(F)$): the trivial representation on the zero dimensional vector space $\{0\}$ and the representation (π, V) itself.

An important finiteness property of smooth representations is the following.

Definition 2.4.2. A smooth representation (π, V) of $G(F)$ is called *admissible* if the space

$$V^K := \{v \in V \,|\, \pi(k)v = v \;\forall\, k \in K\}$$

of K-fixed vectors has finite dimension for every compact open subgroup K of $G(F)$.

An important fact for representations with complex coefficients is that irreducible smooth representations are automatically admissible.

Fact 2.4.3. *If (π, V) is an irreducible smooth representation of $G(F)$, then (π, V) is admissible.*

An important tool to construct representations is the induction. There are two kinds of inductions that will play an important role for us.

Definition 2.4.4. Let H be a closed subgroup of $G(F)$ (with respect to the p-adic topology underlying the topological group $G(F)$) and let (σ, W) be a smooth representation of H.

The *induced representation* $(R, \mathrm{Ind}_H^{G(F)} W)$ (also sometimes referred to as *smooth induction*) is defined as follows:

- $\mathrm{Ind}_H^{G(F)} W$ is the space of functions $f : G(F) \to W$ satisfying
  (a) $f(hg) = \sigma(h)f(g)$ for all $h \in H, g \in G(F)$, and
  (b) there exists a compact open subgroup $K_f \subset G(F)$ such that $f(gk) = f(g)$ for all $k \in K_f$.

- The action of $G(F)$ on $\mathrm{Ind}_H^{G(F)} W$ is via right translation, i.e.

$$(R(g)(f))(x) = f(xg) \ \forall\, g \in G(F), f \in \mathrm{Ind}_H^{G(F)} W, x \in G(F).$$

We may also write $(\mathrm{Ind}_H^{G(F)} \sigma, \mathrm{Ind}_H^{G(F)} W)$ instead of $(R, \mathrm{Ind}_H^{G(F)} W)$.

The *compact induction* of (σ, W) from H to $G(F)$ is the subrepresentation $(R, \text{c-ind}_H^{G(F)} W)$ of $(R, \mathrm{Ind}_H^{G(F)} W)$ consisting of functions $f \in \mathrm{Ind}_H^{G(F)} W$ whose support has compact image in $H\backslash G(F)$. We may also write $(\text{c-ind}_H^{G(F)} \sigma, \text{c-ind}_H^{G(F)} W)$ instead of $(R, \text{c-ind}_H^{G(F)} W)$.

For the smooth induction, we are particularly interested in the following special case.

Definition 2.4.5. Let $P \subset G$ be a parabolic subgroup of G with Levi decomposition $P = M \ltimes N$. Let (σ, W) be a smooth representation of the Levi subgroup $M(F)$. The *parabolic induction* $(\mathrm{Ind}_{P(F)}^{G(F)} \sigma, \mathrm{Ind}_{P(F)}^{G(F)} W)$ is defined by inflating (i.e. extending) the representation σ to a representation of $P(F)$ that is trivial on $N(F)$ and then inducing the resulting representation from $P(F)$ to $G(F)$.

Remark 2.4.6. We caution the reader that some authors normalize the parabolic induction by replacing $\sigma(m)$ by $\sigma(m) \left|\det \mathrm{Ad}_{\mathrm{Lie}(N)(F)}(m)\right|^{1/2}$ for $m \in M(F)$. This normalized induction preserves unitary. However, for our applications, both parabolic inductions, the normalized and the unnormalized one, work equally well.

This allows us to define supercuspidal representations.

Definition 2.4.7. An irreducible smooth representation (π, V) of $G(F)$ is called *supercuspidal* if for all proper parabolic subgroups $P \subsetneq G$ with Levi subgroup M and all irreducible smooth representations (σ, W) of $M(F)$ the representation (π, V) is not a subrepresentation of $(\mathrm{Ind}_{P(F)}^{G(F)} \sigma, \mathrm{Ind}_{P(F)}^{G(F)} W)$.

The following fact explains why we call the supercuspidal representations the building blocks.

Fact 2.4.8. *Let (π, V) be an irreducible smooth representation of G. Then there exists a parabolic subgroup $P \subseteq G$ with Levi subgroup M and a supercuspidal representation (σ, W) of $M(F)$ such that (π, V) is a subrepresentation of $(\mathrm{Ind}_{P(F)}^{G(F)} \sigma, \mathrm{Ind}_{P(F)}^{G(F)} W)$.*

It is a folklore conjecture that all supercuspidal representations arise via compact induction from a representation of a compact-mod-center open subgroup. In this survey we will outline how to construct all supercuspidal representations via compact induction under some mild tameness assumptions. In order to do this, we will need to introduce some additional structure theory. Though before doing so in the next section, let us mention the analogous definition of supercuspidal representations in the finite group case for later use.

Definition 2.4.9. Let H be the $\mathbb{F}_q$-points of a reductive group. An irreducible representation (π, V) of H is called *cuspidal* if the following equivalent conditions are satisfied:

(a) There does not exist a proper parabolic subgroup $P = MN$ of H and a representation (σ, W) of a Levi subgroup M such that V embeds into the induced representation $(\mathrm{Ind}_P^H \sigma, \mathrm{Ind}_P^H W)$.

(b) There does not exist a proper parabolic subgroup P of H with unipotent radical N such that the space of N-fixed vectors V^N is non-trivial.

We conclude this section by stating an equivalent definition of supercuspidal representations, for which we first introduce the contragredient representation.

Definition 2.4.10. Let (π, V) be a smooth representation of $G(F)$. We denote by V^* the dual vector space of V with the (often not smooth) representation π^* given by

$$\pi^*(g)(\lambda)(v) = \lambda(\pi(g^{-1})v) \quad \text{for } g \in G(F), \lambda \in V^*, v \in V.$$

The *contragredient representation* $(\widetilde{\pi}, \widetilde{V})$ is the restriction of the representation (π^*, V^*) to the subspace of smooth vectors $\widetilde{V} := \bigcup_K (V^*)^K$, where the union runs over all compact open subgroups K of $G(F)$.

Fact 2.4.11. *An irreducible smooth representations (π, V) of $G(F)$ is supercuspidal if and only if the image in $G(F)/Z(G(F))$ of the support of the function*

$$\begin{array}{rcl} G(F) & \to & \mathbb{C} \\ g & \mapsto & \lambda(\pi(g)v) \end{array}$$

is compact for all $v \in V, \lambda \in \widetilde{V}$, where $Z(G(F))$ denotes the center of $G(F)$. Equivalently, we may ask this condition to be satisfied only for some $0 \neq v \in V$ and $0 \neq \lambda \in \widetilde{V}$.

3 An introduction to the Moy–Prasad filtration and Bruhat–Tits theory

The Moy–Prasad filtration is a decreasing filtration of $G(F)$ by compact open subgroups that are normal inside each other and whose intersection is trivial. It is a refinement and generalization of the congruence filtration of $\mathrm{GL}_n(F)$. One usually starts with the definition of a Bruhat–Tits building that Bruhat and Tits ([BT72, BT84]) attached to the reductive group G over F in 1972/1984, and then to each point in the Bruhat–Tits building, Moy and Prasad ([MP94, MP96]) associated in 1994/1996 a filtration by compact open subgroups. In this survey, we will take a different approach and first introduce the Moy–Prasad filtration and use it to define the Bruhat–Tits building.

We abbreviate the ring of integers $\mathcal{O}_F$ of F by $\mathcal{O}$ and write $\mathbb{F}_q$ for the *residue field*, which is defined to be the quotient $\mathcal{O}/\mathcal{P}_F$. The residue field $\mathbb{F}_q$ is a finite field with q elements for some power q of p. We fix a uniformizer $\varpi \in \mathcal{P}_F$, and we define the valuation $\mathrm{val} : F \twoheadrightarrow \mathbb{Z} \cup \{\infty\}$ by

$$\mathrm{val}(x) = \frac{\log(|x|_p)}{\log(|\varpi|_p)} \;\; \forall x \in F \setminus \{0\} \;\text{ and }\; \mathrm{val}(0) = \infty.$$

Then

$$\mathrm{val}(\varpi) = 1 \text{ and } \mathcal{O} = \{x \in F \mid \mathrm{val}(x) \geq 0\}.$$

We extend this valuation to any finite field extension E of F using the same formula. The valuation on E takes values in $\mathbb{Q} \cup \{\infty\}$.

3.1 The split case

We assume in this subsection that G is split over F. Let T be a split maximal torus of G. We recall that a *Chevalley system* $\{X_\alpha\}_{\alpha \in \Phi(G,T)}$ consists of a non-trivial element X_α in the one dimensional F-vector space $\mathfrak{g}_\alpha(F)$ for each root α of G with respect to T such that

$$\mathrm{Ad}(w_\beta)(X_\alpha) = \pm X_{s_\beta(\alpha)} \,, \; \forall \alpha, \beta \in \Phi(G,T),$$

where w_β is an element of the normalizer $N(T)(F)$ of T determined by X_β whose image in the Weyl group $(N(T)/T)(F)$ is the simple reflection s_β corresponding to β. For example, if $G = \mathrm{SL}_2$ and $X_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $w_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In general w_β is defined as follows. For every root β, we let $x_\beta : \mathbb{G}_a \xrightarrow{\simeq} U_\beta$ be the isomorphism whose derivative sends $1 \in F = \mathbb{G}_a(F)$ to X_β, then

$$w_\beta = x_\beta(1) x_\beta(\epsilon) x_\beta(1)$$

where $\epsilon \in \{\pm 1\}$ is the unique element for which $x_\beta(1) x_\beta(\epsilon) x_\beta(1)$ lies in the normalizer of T.

For example, for GL_n the collection $\{X_{\alpha_{i,j}}\}_{1 \leq i,j \leq n; i \neq j}$ consisting of the matrices with all entries zero except for a one at position (i, j) forms a Chevalley system.

This allows us to make the following definition, but we warn the reader that we have not seen anyone else use the terminology "BT triple".

Notation 3.1.1. A *BT triple* (T, X_α, x_{BT}) consists of

(i) a split maximal torus T of G,

(ii) a Chevalley system $\{X_\alpha\}_{\alpha\in\Phi(G,T)}$, and

(iii) $x_{BT} \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} := \mathrm{Hom}_F(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R}$.

Here Hom_F denotes homomorphisms in the category of F-group schemes, i.e. $\mathrm{Hom}_F(\mathbb{G}_m, T)$ denotes the homomorphisms between the F-varieties $\mathbb{G}_m$ and T that commute with the group action. Then $\mathrm{Hom}_F(\mathbb{G}_m, T)$ is a free $\mathbb{Z}$-module, hence $\mathrm{Hom}_F(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R}$ is a finite-dimensional real vector space. Moreover, we have a bilinear pairing between $X^*(T) := \mathrm{Hom}_F(T, \mathbb{G}_m)$ and $X_*(T) = \mathrm{Hom}_F(\mathbb{G}_m, T)$ obtained by identifying $\mathrm{Hom}_F(\mathbb{G}_m, \mathbb{G}_m)$ with $\mathbb{Z}$. We extend this map $\mathbb{R}$-linearly in the second factor to obtain a map

$$\langle\cdot,\cdot\rangle : X^*(T) \times X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}.$$

In particular, we may pair x_{BT} with a root $\alpha \in \Phi(G,T)$ to obtain a real number $\langle \alpha, x_{BT}\rangle$.

We now fix a BT triple $x = (T, \{X_\alpha\}, x_{BT})$ and define the Moy–Prasad filtration attached to it.

Filtration of the torus.

We set

$$T(F)_0 = \{t \in T(F) \mid \mathrm{val}(\chi(t)) = 0 \ \forall\, \chi \in X^*(T) = \mathrm{Hom}_F(T, \mathbb{G}_m)\},$$

which is the maximal bounded subgroup of $T(F)$. For $r \in \mathbb{R}_{>0}$, we define

$$T(F)_r = \{t \in T(F)_0 \mid \mathrm{val}(\chi(t) - 1) \geq r \ \forall\, \chi \in X^*(T)\}.$$

For example, if $G = \mathrm{GL}_n$ and T is the torus consisting of diagonal matrices, then $T(F)_0$ consists of diagonal matrices whose entries are all in $\mathcal{O}^\times$ and $T(F)_r$ consists of diagonal matrices whose entries are all in $1 + \varpi^{\lceil r \rceil}\mathcal{O}$.

Filtration of the root groups.

Let $\alpha \in \Phi(G,T)$. We recall that the isomorphism $x_\alpha : \mathbb{G}_a \to U_\alpha$ is defined by requiring its derivative dx_α to send $1 \in F = \mathbb{G}_a(F)$ to X_α. For $r \in \mathbb{R}_{\geq 0}$, we define the filtration subgroups of $U_\alpha(F)$ as follows

$$U_\alpha(F)_{x,r} := x_\alpha(\varpi^{\lceil r - \langle \alpha, x_{BT}\rangle \rceil}\mathcal{O}).$$

Let us consider the example of $G = \mathrm{SL}_2$ and T the torus consisting of diagonal matrices.

Example 1. Let x_1 be the Bruhat–Tits triple $\left(T, \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}, 0\right)$. Let $\alpha = \alpha_{1,2}$, and hence $-\alpha = \alpha_{2,1}$. Then $x_\alpha(y) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ for $y \in F = \mathbb{G}_a(F)$ and

$$U_\alpha(F)_{x_1,r} = \begin{pmatrix} 1 & \varpi^{\lceil r \rceil}\mathcal{O} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U_{-\alpha}(F)_{x_1,r} = \begin{pmatrix} 1 & 0 \\ \varpi^{\lceil r \rceil}\mathcal{O} & 1 \end{pmatrix}.$$

Example 2. Let x_2 be the Bruhat–Tits triple $\left(T, \left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}, \frac{1}{4}\check{\alpha}\right)$, where $\check{\alpha}$ is the coroot of α, i.e. the element of $X_*(T)$ that satisfies $\check{\alpha}(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ for $t \in F^\times = \mathbb{G}_m(F)$. Then

$$U_\alpha(F)_{x_2,r} = \begin{pmatrix} 1 & \varpi^{\lceil r-\frac{1}{2}\rceil}\mathcal{O} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U_{-\alpha}(F)_{x_2,r} = \begin{pmatrix} 1 & 0 \\ \varpi^{\lceil r+\frac{1}{2}\rceil}\mathcal{O} & 1 \end{pmatrix}.$$

Filtration of $G(F)$.
We define the filtration subgroup $G(F)_{x,r}$ of $G(F)$ for $r \in \mathbb{R}_{\geq 0}$ to be the subgroup generated by $T(F)_r$ and $U_\alpha(F)_{x,r}$ for all roots α, i.e.

$$G(F)_{x,r} = \langle T(F)_r, U_\alpha(F)_{x,r} \mid \alpha \in \Phi(G,T) \rangle .$$

If the ground field F is clear from the context, we may also abbreviate $G(F)_{x,r}$ by $G_{x,r}$.

In the example of $G = \mathrm{SL}_2$ for the two Bruhat–Tits triples above, we have for $r > 0$

$$G_{x_1,0} = \mathrm{SL}_2(\mathcal{O})\,, \quad G_{x_1,r} = \begin{pmatrix} 1+\varpi^{\lceil r\rceil}\mathcal{O} & \varpi^{\lceil r\rceil}\mathcal{O} \\ \varpi^{\lceil r\rceil}\mathcal{O} & 1+\varpi^{\lceil r\rceil}\mathcal{O} \end{pmatrix}_{\det=1}$$

and

$$G_{x_2,0} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O} \end{pmatrix}_{\det=1}, \quad G_{x_2,r} = \begin{pmatrix} 1+\varpi^{\lceil r\rceil}\mathcal{O} & \varpi^{\lceil r-\frac{1}{2}\rceil}\mathcal{O} \\ \varpi^{\lceil r+\frac{1}{2}\rceil}\mathcal{O} & 1+\varpi^{\lceil r\rceil}\mathcal{O} \end{pmatrix}_{\det=1}.$$

Filtration of $\mathfrak{g}(F)$ and $\mathfrak{g}^*(F)$.
One can analogously define a filtration $\mathfrak{g}_{x,r}$ of the Lie algebra $\mathfrak{g}(F)$ and a filtration $\mathfrak{g}^*_{x,r}$ of the F-linear dual $\mathfrak{g}^*(F)$ of the Lie algebra $\mathfrak{g}(F)$ as follows. Let r be a real number, and recall that we write $\mathfrak{t}$ for the Lie algebra of the torus T. Then we set

$$\mathfrak{t}(F)_r = \{X \in \mathfrak{t}(F) \mid \mathrm{val}(d\chi(X)) \geq r \;\forall\, \chi \in X^*(T)\},$$

where $d\chi$ denotes the derivative of χ,

$$\mathfrak{g}_\alpha(F)_{x,r} = \varpi^{\lceil r-\langle \alpha, x_{BT}\rangle\rceil}\mathcal{O}\, X_\alpha \subset \mathfrak{g}_\alpha(F)$$

for $\alpha \in \Phi(G,T)$, and

$$\mathfrak{g}(F)_{x,r} = \mathfrak{t}(F)_r \oplus \bigoplus_{\alpha\in\Phi(G,T)} \mathfrak{g}_\alpha(F)_{x,r}.$$

We define the filtration subspace $\mathfrak{g}^*(F)_{x,r}$ of the dual of the Lie algebra by

$$\mathfrak{g}^*(F)_{x,r} = \{X \in \mathfrak{g}^*(F) \mid X(Y) \in \varpi\mathcal{O} \text{ for all } Y \in \mathfrak{g}(F)_{x,s} \text{ with } s > -r\}.$$

If the ground field F is clear from the context, we may also abbreviate $\mathfrak{g}(F)_{x,r}$ and $\mathfrak{g}^*(F)_{x,r}$ by $\mathfrak{g}_{x,r}$ and $\mathfrak{g}^*_{x,r}$, respectively.

Properties of the Moy–Prasad filtration

Definition 3.1.2. A *parahoric* subgroup of G is a subgroup of the form $G_{x,0}$ for some BT triple x.

For $r \in \mathbb{R}_{\geq 0}$, we write $G_{x,r+} = \bigcup_{s>r} G_{x,s}$ and $\mathfrak{g}_{x,r+} = \bigcup_{s>r} \mathfrak{g}_{x,s}$.
We collect a few facts about this filtration.

Fact 3.1.3. *Let x be a BT triple.*

(a) $G_{x,r}$ is a normal subgroup of $G_{x,0}$ for all $r \in \mathbb{R}_{\geq 0}$.

(b) The quotient $G_{x,0}/G_{x,0+}$ is the group of $\mathbb{F}_q$-points of a reductive group $\mathbb{G}_x$ defined over the residue field $\mathbb{F}_q$ of F.

(c) For $r \in \mathbb{R}_{>0}$, the quotient $G_{x,r}/G_{x,r+}$ is abelian and can be identified with an $\mathbb{F}_q$-vector space.

(d) Let $r > 0$. Since $G_{x,r}$ is a normal subgroup of $G_{x,0}$, the group $G_{x,0}$ acts on $G_{x,r}$ via conjugation. This action descends to an action of the quotient $G_{x,0}/G_{x,0+}$ on the vector space $G_{x,r}/G_{x,r+}$ and the resulting action is (the $\mathbb{F}_q$-points of) a linear algebraic action, i.e. corresponds to a morphism from $\mathbb{G}_x$ to $\mathrm{GL}_{\dim_{\mathbb{F}_q}(G_{x,r}/G_{x,r+})}$ over $\mathbb{F}_q$.

(e) We have the following isomorphism that is often referred to as the "Moy–Prasad isomorphism": $G_{x,r}/G_{x,r+} \simeq \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$ for $r \in \mathbb{R}_{>0}$ and more general $G_{x,r}/G_{x,s} \simeq \mathfrak{g}_{x,r}/\mathfrak{g}_{x,s}$ for $r, s \in \mathbb{R}_{>0}$ with $2s \geq r$.

In fact we have a rather good understanding of the representations occurring in (d). In [Fin21b] they are described explicitly in terms of Weyl modules. Previously they were also realized using Vingberg–Levy theory by Reeder and Yu ([RY14]), which was generalized in [Fin21b].

The Bruhat–Tits building

Definition 3.1.4. The *(reduced) Bruhat–Tits building* $\mathscr{B}(G,F)$ of G over F is as a set the quotient of the set of BT triples by the following equivalence relation: Two BT triple x_1 and x_2 are equivalent if and only if $G_{x_1,r} = G_{x_2,r}$ for all $r \in \mathbb{R}_{\geq 0}$.

As a consequence of the definition, for $x \in \mathscr{B}(G,T)$, we may write $G_{x,r}$ for the Moy–Prasad filtration attached to any BT triple in the equivalence class of x.

The Bruhat–Tits building $\mathscr{B}(G,F)$ admits an action of $G(F)$ that is determined by the property
$$G_{g.x,r} = gG_{x,r}g^{-1} \; \forall \, r \in \mathbb{R}_{\geq 0}, g \in G(F).$$

We will now equip the Bruhat–Tits building with more structure.

Apartments as affine spaces.

Definition 3.1.5. For a split maximal torus T, we call the subset of $\mathscr{B}(G,F)$ that can be represented by BT triples whose first entry is the given torus T, i.e.

$$\mathscr{A}(T,F) := \{(T, \{X_\alpha\}, x_{BT})\}/_\sim \ \subset \mathscr{B}(G,F)$$

the *apartment* of T.

We fix a split maximal torus T and a Chevalley system $\{X_\alpha\}_{\alpha\in\Phi(G,T)}$. Then it turns out that every element in $\mathscr{A}(T,F)$ can be represented by a BT triple whose first two entries are the torus T and the fixed Chevalley system $\{X_\alpha\}_{\alpha\in\Phi(G,T)}$. Moreover, two BT triples $(T,\{X_\alpha\},x_{BT,1})$ and $(T,\{X_\alpha\},x_{BT,2})$ are equivalent if and only if $x_{BT,2}-x_{BT,1}$ lies in the subspace $X_*(Z(G))\otimes\mathbb{R}$, where $Z(G)$ denotes the center of G. Note that $X_*(Z(G))\otimes\mathbb{R}$ is trivial when the center $Z(G)$ of G is finite. Thus the set $\mathscr{A}(T,F)$ is isomorphic to $X_*(T)\otimes\mathbb{R}/X_*(Z(G))\otimes\mathbb{R}$, and we use this isomorphism to equipp $\mathscr{A}(T,F)$ with the structure of an affine space over the real vector space $X_*(T)\otimes\mathbb{R}/X_*(Z(G))\otimes\mathbb{R}$. While the isomorphism of $\mathscr{A}(T,F)$ with $X_*(T)\otimes\mathbb{R}/X_*(Z(G))\otimes\mathbb{R}$ depends on the choice of the Chevalley system $\{X_\alpha\}_{\alpha\in\Phi(G,T)}$, the structure of $\mathscr{A}(T,F)$ as an affine space does not. In fact, the choice of a Chevalley system turns the affine space into a vector space by choosing a base point.

Polysimplicial structure on apartments.

Let T be a split maximal torus of G. For $\alpha\in\Phi(G,T)$, we define the following set of hyperplanes of the apartment $\mathscr{A}(T,F)$:

$$\Psi_\alpha := \left\{ \text{hyperplanes } H\subset\mathscr{A}(T,F) \text{ satisfying } \begin{array}{ll} U_\alpha(F)_{x,0}=U_\alpha(F)_{y,0} & \forall x,y\in H \\ U_\alpha(F)_{x,0}\neq U_\alpha(F)_{x,0+} & \forall x\in H \end{array} \right\}.$$

We set

$$\Psi := \bigcup_{\alpha\in\Phi(G,T)} \Psi_\alpha$$

and use these hyperplanes to turn the apartment $\mathscr{A}(T,F)$ into the geometric realization of a polysimplicial complex. This means the connected components of the complement of the union of the hyperplanes in Ψ are the maximal dimensional polysimplices, which are also called *chambers*.

We record the following facts that will become useful later when constructing supercuspidal representations.

Fact 3.1.6. *Let $x\in\mathscr{A}(T,F)\subset\mathscr{B}(G,F)$.*

(a) The root system of $\mathbb{G}_x$ is given by $\Phi(\mathbb{G}_x)=\{\alpha\in\Phi(G,T)\,|\,x\in H$ for some $H\in\Psi_\alpha\}$.

(b) Let $y\in\mathscr{A}(T,F)$. Then the image of $G_{x,0}\cap G_{y,0}$ in $G_{y,0}/G_{y,0+}$ is a parabolic subgroup $P_{x,y}$ and the image of $G_{x,0+}\cap G_{y,0}$ in $G_{y,0}/G_{y,0+}$ is the unipotent radical of $P_{x,y}$. If $x\neq y$ and y is a vertex, i.e. a polysimplex of minimal dimension, then $P_{x,y}$ is a proper parabolic subgroup.

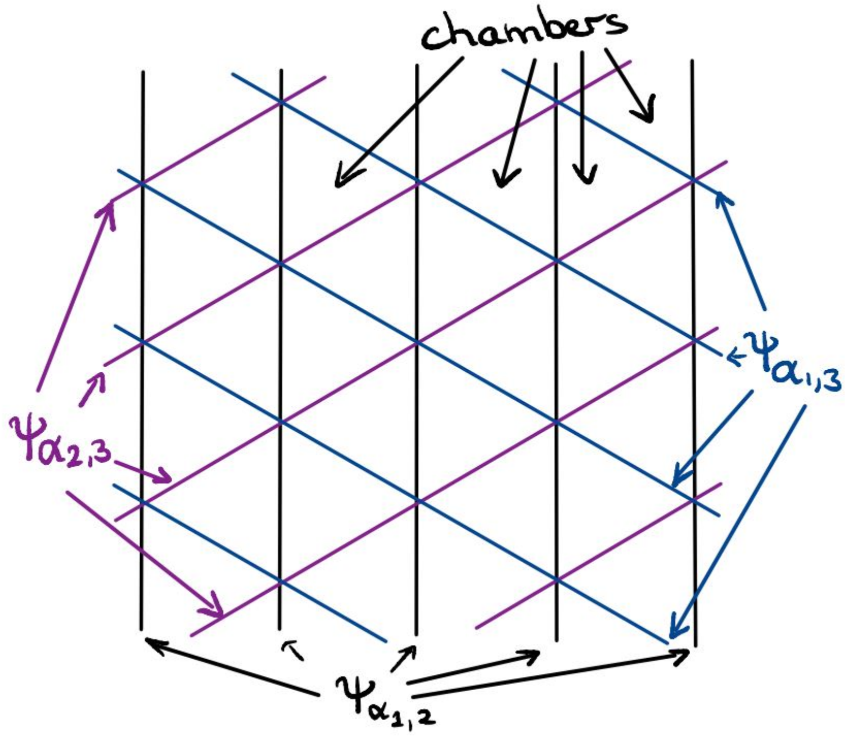

Figure 5: Excerpt of an apartment for SL_3 with hyperplanes

3.2 The non-split (tame) case

We first assume that G splits over an unramified Galois field extension E over F. In that case all the above definitions can be descended to G by taking $\mathrm{Gal}(E/F)$-fixed points of the corresponding objects for G_E. More precisely, we set

$$G(F)_{x,r} = G(E)_{x,r}^{\mathrm{Gal}(E/F)},$$

where $G(E)_{x,r}$ is defined using the valuation on E that extends the valuation val on F. As in the split case, we may abbreviate $G(F)_{x,r}$ by $G_{x,r}$.

Via the action of $\mathrm{Gal}(E/F)$ on $G(E)$ and hence on its filtration subgroups, we obtain an action of $\mathrm{Gal}(E/F)$ on the Bruhat–Tits building $\mathscr{B}(G,E)$ and we define

$$\mathscr{B}(G,F) = \mathscr{B}(G,E)^{\mathrm{Gal}(E/F)}.$$

More generally, if we only assume that G splits over a tamely ramified Galois field extension E over F, then we have for $r > 0$

$$G_{x,r} = G(F)_{x,r} = G(E)_{x,r}^{\mathrm{Gal}(E/F)},$$

where $G(E)_{x,r}$ is defined using the valuation on E that extends the valuation val on F and $U_\alpha(E)_{x,r} = x_\alpha(\varpi_E^{\lceil e(r-\langle \alpha, x_{BT}\rangle)\rceil}\mathcal{O}_E)$ with e the ramification index of the field extension E

over F. Defining the parahoric subgroup $G(F)_{x,0}$ is slightly more subtle in general. It is a finite index subgroup of $G(E)_{x,0}^{\mathrm{Gal}(E/F)}$. The parahoric subgroup $G(F)_{x,0}$ being occasionally a slightly smaller group than $G(E)_{x,0}^{\mathrm{Gal}(E/F)}$ will ensure that $G(F)_{x,0}/G(F)_{x,0+}$ are the $\mathbb{F}_q$-points of a connected reductive group rather than a potentially disconnected group. More precisely, the parahoric subgroup $G(F)_{x,0}$ is defined by

$$G_{x,0} = G(F)_{x,r} = G(E)_{x,r}^{\mathrm{Gal}(E/F)} \cap G(F)^0$$

for some explicitly constructed normal subgroup $G(F)^0 \subset G(F)$. We refer the interested reader to the literature, e.g. [KP], for the precise definition of $G(F)^0$ and only note that $G(F)^0 = G(F)$ if G is simply connected semi-simple, e.g. for $G = \mathrm{SL}_n$ we have $\mathrm{SL}_n(F)^0 = \mathrm{SL}_n(F)$.

As in the unramified setting, using the action of $\mathrm{Gal}(E/F)$ on $G(E)$ and hence on its filtration subgroups, we obtain an action of $\mathrm{Gal}(E/F)$ on the Bruhat–Tits building $\mathscr{B}(G,E)$ and we define

$$\mathscr{B}(G,F) = \mathscr{B}(G,E)^{\mathrm{Gal}(E/F)}.$$

Similarly, we have for the Lie algebra

$$\mathfrak{g}_{x,r} = \mathfrak{g}(F)_{x,r} = (\mathfrak{g}(E)_{x,r})^{\mathrm{Gal}(E/F)}.$$

We note that the above definitions rely on the extension E over F being tame, but are independent of the choice of E.

Aside 3.2.1. If G splits only over a wildly ramified extension E/F, then the space of fixed vectors of the Galois action on the Bruhat–Tits building over E might be larger than the Bruhat–Tits building defined over F (which we have not introduced in that generality in this survey).

3.3 The enlarged Bruhat–Tits building

In some circumstances it is more convenient to work with the enlarged Bruhat–Tits building. The *enlarged Bruhat–Tits building* $\widetilde{\mathscr{B}}(G,F)$ is defined as the product of the reduced Bruhat–Tits building $\mathscr{B}(G,F)$ with $X_*(Z(G)) \otimes_{\mathbb{Z}} \mathbb{R}$, i.e.

$$\widetilde{\mathscr{B}}(G,F) = \mathscr{B}(G,F) \times X_*(Z(G)) \otimes_{\mathbb{Z}} \mathbb{R}.$$

This means that if the center of G is finite, then the reduced and the non-reduced Bruhat–Tits buildings are the same. In general, an important difference is that stabilizers in $G(F)$ of points in the enlarged Bruhat–Tits building are compact while stabilizers of points in the reduced Bruhat–Tits building contain the center of $G(F)$ and are compact-mod-center. For the enlarged building, the apartments $\widetilde{\mathscr{A}}(S,F)$ correspond to maximal split tori S and are affine spaces under the action of $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. For a point $x \in \widetilde{\mathscr{B}}(G,F)$ we denote by $[x]$ the image of x in $\mathscr{B}(G,F)$ (by projection on the first factor) and we define $G_{x,r} := G_{[x],r}$ for $r \in \mathbb{R}_{\geq 0}$ and $\mathfrak{g}_{x,r} := \mathfrak{g}_{[x],r}$ and $\mathfrak{g}^*_{x,r} := \mathfrak{g}^*_{[x],r}$ for $r \in \mathbb{R}$.

3.4 The depth of a representation

The Moy–Prasad filtration allows us to introduce the notion of the depth of a representation, initially defined by Moy and Prasad in [MP94, MP96]. Our definition is slightly different but equivalent to theirs.

Definition 3.4.1. Let (π, V) be an irreducible smooth representation of G. The *depth* of (π, V) is the smallest non-negative real number r such that $V^{G_{x,r+}} \neq \{0\}$ for some $x \in \mathscr{B}(G, F)$.

4 Construction of supercuspidal representations

As discussed in the introduction, mathematicians have worked on the construction of supercuspidal representations the past 50 years. Here we will present the construction of supercuspidal representations by Yu ([Yu01]) from 2001, with a twist introduced by Fintzen, Kaletha and Spice ([FKS]) in 2021. Contrary to earlier works, this construction applies to all p-adic groups that split over a tamely ramified extension and is exhaustive if p does not divide the order of the (absolute) Weyl group of the p-adic group. Yu's construction is a generalization of a construction by Adler ([Adl98]), which in term was inspired by work of Howe, Morris, Moy and unpublished work of Jabon.

4.1 The input for the construction

We fix for the rest of the paper an additive character $\varphi : F \to \mathbb{C}^\times$ (i.e. a group homomorphism from the group F (equipped with addition) to the group $\mathbb{C}^\times$ (equipped with multiplication)) that is nontrivial on $\mathcal{O}$ and trivial on $\varpi\mathcal{O}$ and we assume that $p \neq 2$.

Definition 4.1.1. A subgroup G' of G is a *twisted Levi subgroup* if G'_E is a Levi subgroup of G_E for some finite field extension E over F.

If G' is a twisted Levi subgroup of G, and we assume that G' splits over a tamely ramified field extension of F, then we have an embedding of the enlarged Bruhat–Tits building $\widetilde{\mathscr{B}}(G', F)$ of G' into the enlarged Bruhat–Tits building $\widetilde{\mathscr{B}}(G, F)$ of G. This embedding is unique up to translation by $X_*(Z(G')) \otimes_{\mathbb{Z}} \mathbb{R}$. Below we will fix such embeddings when working with twisted Levi subgroups to view $\widetilde{\mathscr{B}}(G', F)$ as a subset of $\widetilde{\mathscr{B}}(G, F)$.

The input for the construction of supercuspidal representations by Yu (following the notation of [Fin21a]) is a tuple $((G_i)_{1\leq i\leq n+1}, x, (r_i)_{1\leq i\leq n}, \rho, (\phi_i)_{1\leq i\leq n})$ for some non-negative integer n where

(a) $G = G_1 \supseteq G_2 \supsetneq G_3 \supsetneq \ldots \supsetneq G_{n+1}$ are twisted Levi subgroups of G that split over a tamely ramified extension of F,

(b) $x \in \widetilde{\mathscr{B}}(G_{n+1}, F) \subset \widetilde{\mathscr{B}}(G, F)$,

(c) $r_1 > r_2 > \ldots > r_n > 0$ are real numbers,

(d) ρ is an irreducible representation of $(G_{n+1})_{[x]}$ that is trivial on $(G_{n+1})_{x,0+}$,

(e) ϕ_i, for $1 \leq i \leq n$, is a character (i.e. a one-dimensional representation) of $G_{i+1}(F)$ of depth r_i

satisfying the following conditions

(i) $Z(G_{n+1})/Z(G)$ is anisotropic, i.e. its F-points are a compact group,

(ii) the image $[x]$ of the point x in $\mathscr{B}(G_{n+1}, F)$ is a vertex, i.e. a polysimplex of minimal dimension

(iii) $\rho|_{(G_{n+1})_{x,0}}$ is a cuspidal representation of the reductive group $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$,

(iv) ϕ_i is (G_i, G_{i+1})-generic relative to x of depth r_i for all $1 \leq i \leq n$ with $G_i \neq G_{i+1}$,

where generic characters are defined below. We will call a tuple satisfying the above conditions a *Yu datum*.

Aside 4.1.2. Our conventions for the notation (following [Fin21a]) differ slightly from those in [Yu01]. In particular, Yu's notation for the twisted Levi sequence is $G^0 \subsetneq G^1 \subsetneq G^2 \subsetneq \ldots \subsetneq G^d$. The reader can find a translation between the two different notations in [Fin21a, Remark 2.4].

In order to define generic character (following [Fin, §2.1], which is based on [Yu01, §9], but is slightly more general for small primes), we first define the notion of generic elements in the dual of the Lie algebra and then use the Moy–Prasad isomorphism to obtain the notion of generic characters.

Let $G' \subsetneq G$ be a twisted Levi subgroup that splits over a tamely ramified extension of F, and denote by $(\mathrm{Lie}^*(G'))^{G'}(F)$ the subspace of the linear dual of $\mathrm{Lie}(G')(F)$ that is fixed by (the dual of) the adjoint action of $G'(\bar{F})$.

Definition 4.1.3. Let $x \in \widetilde{\mathscr{B}}(G', F)$ and $r \in \mathbb{R}_{>0}$.

(a) An element X of $(\mathrm{Lie}^*(G'))^{G'}(F) \subset \mathrm{Lie}^*(G')(F)$ is called *G-generic of depth* r (or *(G, G')-generic of depth* r) if the following three conditions hold.

(GE0) For some (equivalently, every) point $x \in \widetilde{\mathscr{B}}(G', F)$, we have $X \in \mathrm{Lie}^*(G')_{x,-r}$.

(GE1) $\mathrm{val}(X(H_\alpha)) = -r$ for all $\alpha \in \Phi(G_{\bar{F}}, T_{\bar{F}}) \backslash \Phi(G'_{\bar{F}}, T_{\bar{F}})$ for some maximal torus T of G', where $H_\alpha := d\check{\alpha}(1) \in \mathfrak{g}(\bar{F})$ with $d\check{\alpha}$ the derivative of the coroot $\check{\alpha} \in \mathrm{Hom}_{\bar{F}}(\mathbb{G}_m, T_{\bar{F}})$ of α.

(GE2) **(GE2)** of [Yu01, §8] holds, where we refer the reader to [Yu01] for details. Condition **(GE1)** implies **(GE2)** if p is not a torsion prime for the dual root datum of G, i.e., in particular, if p does not divide the order of the (absolute) Weyl group of G. Hence, by assuming that p is large enough, the reader may ignore Condition (GE2).

(b) A character ϕ of $G'(F)$ is called *G-generic (or (G, G')-generic) relative to x of depth r* if ϕ is trivial on $G'_{x,r+}$, non-trivial on $G'_{x,r}$ and the restriction of ϕ to $G'_{x,r}/G'_{x,r+} \simeq \mathfrak{g}'_{x,r}/\mathfrak{g}'_{x,r+}$ is given by $\varphi \circ X$ for some (G, G')-generic element X of depth r.

For example, if $F = \mathbb{Q}_{17}$, $G = \mathrm{GL}_2$ and G' is the diagonal torus T. Then

$$H_{\alpha_{1,2}} = -H_{\alpha_{2,1}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the elements

$$X : \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto A \quad \text{and} \quad X' : \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto A - B$$

are G-generic of depth 0. The elements

$$X : \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto A + B \quad \text{and} \quad X' : \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto A - 16B$$

are also contained in $\mathrm{Lie}^*(T)_{x,0} \setminus \mathrm{Lie}^*(T)_{x,0+}$ (for any $x \in \widetilde{\mathscr{B}}(T, F)$), but are not G-generic of depth r for any real number r.

4.2 The construction of supercuspidal representations à la Yu

In this section we outline how Yu ([Yu01]) constructs from a Yu datum

$$((G_i)_{1 \le i \le n+1}, x, (r_i)_{1 \le i \le n}, \rho, (\phi_i)_{1 \le i \le n})$$

a compact-mod-center open subgroup $\widetilde{K}$ and a representation $\widetilde{\rho}$ of $\widetilde{K}$ such that $\text{c-ind}_{\widetilde{K}}^{G(F)} \widetilde{\rho}$ is an irreducible supercuspidal representation of $G(F)$.

The compact-mod-center open subgroup $\widetilde{K}$ is given by

$$\widetilde{K} = (G_1)_{x,\frac{r_1}{2}} (G_2)_{x,\frac{r_2}{2}} \dots (G_n)_{x,\frac{r_n}{2}} (G_{n+1})_{[x]},$$

where $(G_{n+1})_{[x]}$ denotes the stabilizer in $G_{n+1}(F)$ of the point $[x]$ in the (reduced) Bruhat–Tits building $\mathscr{B}(G_{n+1}, F)$.

The representation $\widetilde{\rho}$ is a tensor product of two representations ρ and κ,

$$\widetilde{\rho} = \rho \otimes \kappa,$$

where ρ also denotes the extension of the representation ρ of $(G_{n+1})_{[x]}$ to $\widetilde{K}$ that is trivial on $(G_1)_{x,\frac{r_1}{2}}(G_2)_{x,\frac{r_2}{2}}\dots(G_n)_{x,\frac{r_n}{2}}$. The representation κ is built out of the characters $\phi_1,\dots,\phi_n$. If $n=0$, then κ is trivial and we are in the setting of depth-zero representations.

We will first sketch the construction of κ in the case $n=1$, i.e. when the Yu datum is of the form $((G=G_1\supset G_2=G_{n+1}),x,(r_1),\rho,(\phi_1))$. To simplify notation, we write $r=r_1$ and $\phi=\phi_1$, and we assume $G_1\neq G_2$. In this case $\widetilde{K}=G_{x,\frac{r}{2}}(G_2)_{[x]}$.

Step 1 (extending the character ϕ as far as possible): The first step consists of extending the character ϕ to a character $\hat{\phi}$ of $G_{x,\frac{r}{2}+}(G_2)_{[x]}$. This is done by sending the root groups outside G_2 to 1. More precisely, $\hat{\phi}$ is the unique character of $G_{x,\frac{r}{2}+}(G_2)_{[x]}$ that satisfies

- $\hat{\phi}|_{(G_2)_{[x]}}=\phi$, and
- $\hat{\phi}|_{G_{x,\frac{r}{2}+}}$ factors through

$$\begin{aligned} G_{x,\frac{r}{2}+}/G_{x,r+} &\simeq \mathfrak{g}_{x,\frac{r}{2}+}/\mathfrak{g}_{x,r+}=(\mathfrak{g}_2(F)\oplus\mathfrak{r}(F))_{x,\frac{r}{2}+}/(\mathfrak{g}_2(F)\oplus\mathfrak{r}(F))_{x,r+}\\ &\to (\mathfrak{g}_2)_{x,\frac{r}{2}+}/(\mathfrak{g}_2)_{x,r+}\simeq(G_2)_{x,\frac{r}{2}+}/(G_2)_{x,r+}, \end{aligned}$$

  on which it is induced by ϕ. Here we used the Moy–Prasad isomorphism and $\mathfrak{r}(F)$ is defined to be

$$\mathfrak{r}(F)=\mathfrak{g}(F)\cap\bigoplus_{\alpha\in\Phi(G_E,T_E)\setminus\Phi((G_2)_E,T_E)}\mathfrak{g}(E)_\alpha$$

  for some maximal torus T of G_2 that splits over a tamely ramified extension E of F with $x\in\widetilde{\mathscr{A}}(T_E,E)$, and the surjection $\mathfrak{g}_2(F)\oplus\mathfrak{r}(F)\twoheadrightarrow\mathfrak{g}_2(F)$ sends $\mathfrak{r}(F)$ to zero.

Step 2 (Heisenberg representation): As second step we extend the (one-dimensional) representation $\hat{\phi}|_{G_{x,\frac{r}{2}+}(G_2)_{x,\frac{r}{2}}}$ to a representation (ω,V_ω) of $G_{x,\frac{r}{2}}$. We write $V_{\frac{r}{2}}$ for the quotient

$$V_{\frac{r}{2}}=G_{x,\frac{r}{2}}/(G_{x,\frac{r}{2}+}(G_2)_{x,\frac{r}{2}})$$

and we view $V_{\frac{r}{2}}$ as an $\mathbb{F}_p$-vector space. (It can also be viewed as an $\mathbb{F}_q$-vector space, but here we only consider the underlying $\mathbb{F}_p$-vector space structure.) Then one can show that the pairing

$$\langle g,h\rangle:=\hat{\phi}(ghg^{-1}h^{-1}),\ g,h\in G_{x,\frac{r}{2}}$$

defines a non-degenerate symplectic form on $V_{\frac{r}{2}}=G_{x,\frac{r}{2}}/(G_{x,\frac{r}{2}+}(G_2)_{x,\frac{r}{2}})$ when we choose an identification between the p-th roots of unity in $\mathbb{C}^\times$ and $\mathbb{F}_p$.

Now the theory of Heisenberg representations implies that there exists a unique irreducible representation (ω,V_ω) of $G_{x,\frac{r}{2}}$ that restricted to $G_{x,\frac{r}{2}+}(G_2)_{x,\frac{r}{2}}$ acts via $\hat{\phi}$ (times identity), and the dimension of V_ω is $\sqrt{\#V_{\frac{r}{2}}}=p^{(\dim_{\mathbb{F}_p}V_{\frac{r}{2}})/2}$.

Step 3 (Weil representation): The final step of the construction consists of extending the action of $G_{x,\frac{r}{2}}$ on V_ω via ω to an action of $\widetilde{K}=G_{x,\frac{r}{2}}(G_2)_{[x]}$ on V_ω by defining an action of

$(G_2)_{[x]}$ on V_ω that is compatible with ω. In order to obtain this action, we first observe that $(G_2)_{[x]}$ acts on $V_{\frac{r}{2}}$ via conjugation and that this action preserves the symplectic form $\langle\cdot,\cdot\rangle$. This provides a morphism from $(G_2)_{[x]}$ to the group $\mathrm{Sp}(V_{\frac{r}{2}})$ of symplectic isomorphisms of $V_{\frac{r}{2}}$. Now the Weil representation is a representation of the symplectic group $\mathrm{Sp}(V_{\frac{r}{2}})$ on the space V_ω of the Heisenberg representation of the symplectic vector space that is compatible with the Heisenberg representation in the following sense. Using the composition of the morphism $(G_2)_{[x]} \to \mathrm{Sp}(V_{\frac{r}{2}})$ with the Weil representation tensored with the character ϕ allows us to extend the representation (ω, V_ω) from $G_{x,\frac{r}{2}}$ to $G_{x,\frac{r}{2}}(G_2)_{[x]}$. We denote the resulting representation of $\widetilde{K} = G_{x,\frac{r}{2}}(G_2)_{[x]}$ also by (ω, V_ω) and set $(\kappa, V_\kappa) = (\omega, V_\omega)$.

This concludes the construction of κ and hence $\widetilde{\rho} = \rho \otimes \kappa$ in the case of $n = 1$. For a more general Yu datum $((G_i)_{1\le i\le n+1}, x, (r_i)_{1\le i\le n}, \rho, (\phi_i)_{1\le i\le n})$ with $n > 1$ we construct from each character ϕ_i $(1 \le i \le n)$ a representation (ω_i, V_{ω_i}) analogous to the construction of (ω, V_ω) above. Then we define κ to be the tensor product of all those representations, i.e.

$$(\kappa, V_\kappa) = \left(\bigotimes_{1\le i\le n} \omega_i, \bigotimes_{1\le i\le n} V_{\omega_i} \right).$$

For the details we refer the reader to [Fin21a, §2.5], which is based on [Yu01].

Theorem 4.2.1 ([Yu01, Fin21a])**.** *The representation* $\text{c-ind}_{\widetilde{K}}^{G(F)} \widetilde{\rho}$ *is a supercuspidal smooth irreducible representation of* $G(F)$*.*

We will sketch the structure of the proof in the next section.

4.3 The proof that the representations are supercuspidal

It is a nice exercise to deduce from Fact 2.4.11 the following lemma.

Lemma 4.3.1. *If* $\text{c-ind}_{\widetilde{K}}^{G(F)} \widetilde{\rho}$ *is irreducible, then* $\text{c-ind}_{\widetilde{K}}^{G(F)} \widetilde{\rho}$ *is a supercuspidal representation of* $G(F)$*.*

This means it suffices to show that $\text{c-ind}_{\widetilde{K}}^{G(F)} \widetilde{\rho}$ is irreducible, for which the standard approach is via Fact 4.3.3 below. In order to state the fact, we need to introduce some notation.

Let K be a compact-mod-center open subgroup of $G(F)$ that contains the center $Z(G(F))$ and let (σ, W) be a smooth representation of K.

Notation 4.3.2. For $g \in G(F)$, we write ${}^g\sigma$ for the representation of ${}^gK := gKg^{-1}$ satisfying ${}^g\sigma(h) = \sigma(g^{-1}hg)$ for $h \in {}^gK$.

We say that g *intertwines* (σ, W) if $\mathrm{Hom}_{{}^gK\cap K}({}^g\sigma|_{{}^gK\cap K}, \sigma|_{{}^gK\cap K}) \neq \{0\}$.

Fact 4.3.3. *Suppose* $g \in G(F)$ *intertwines* (σ, W) *if and only if* $g \in K$*. Then* $\text{c-ind}_K^{G(F)} \sigma$ *is irreducible.*

This result relies on the Mackey decomposition.

Lemma 4.3.4 (Mackey decomposition)**.** *If K' is a compact-mod-center open subgroup of G, then the restriction of* $\text{c-ind}_K^{G(F)} \sigma$ *to K' decomposes as a representation of K' as follows*

$$(\text{c-ind}_K^{G(F)} \sigma)|_{K'} = \bigoplus_{K'\backslash G(F)/K} \text{Ind}_{{}^gK\cap K'}^{K'} {}^g\sigma|_{{}^gK\cap K'}$$

The proof of the lemma is left as an exercise for the reader.

Sketch of the structure of the proof of Theorem 4.2.1.

In order to prove that $\text{c-ind}_{\widetilde{K}}^{G(F)} \widetilde{\rho}$ is supercuspidal it suffices to prove that it is irreducible by Lemma 4.3.1. First one notes that $\widetilde{\rho}$ itself is irreducible. We assume that an element $g \in G(F)$ intertwines $\widetilde{\rho}$. Now the main task is to show that $g \in \widetilde{K}$ so that we can apply Fact 4.3.3. This is done in two steps.

Step 1. We show recursively that $g \in \widetilde{K} G_{n+1} \widetilde{K}$ using that the characters ϕ_i are generic.

The key part for this step is [Yu01, Theorem 9.4], which in the example of $n = 1$ spelled out above implies the following lemma.

Lemma 4.3.5 ([Yu01])**.** *Suppose that g intertwines $\hat{\phi}|_{G_{x,\frac{r}{2}+}}$. Then $g \in G_{x,\frac{r}{2}} G_2(F) G_{x,\frac{r}{2}}$.*

As mentioned above, this lemma crucially uses the fact that ϕ is (G, G_2)-generic relative to x of depth r (if $G_1 \neq G_2$) and we refer to [Yu01, Theorem 9.4] for the proof.

Step 2. By Step 1 we may assume that $g \in G_{n+1}(F)$. Step 2 consists of showing that then $g \in (G_{n+1})_{[x]}$ using the structure of the Weil–Heisenberg representation and that $\rho|_{(G_{n+1})_{x,0}}$ is cuspidal.

The reader interested in the full details of the proof is encouraged to read [Fin21a, §3], which refers to precise statements in [Yu01] that allow an easy backtracking within [Yu01] if the reader is interested in all the details that make the complete proof. While Section 3 of [Fin21a] is only about four pages long, we do not see a merit in copy+pasting it here. Instead we present readers who are only interested in a glimpse of an idea of the proof of Step 2 with the proof in the depth-zero case, i.e. the $n = 0$ case, in this survey. This case has been known already much earlier ([MP96, Mor99]) and does not require an intricate study of the Weil–Heisenberg representations, but on the other hand shows the importance played by $\rho|_{(G_{n+1})_{x,0}}$ being cuspidal.

4.4 Depth-zero supercuspidal representations

In this section, we consider the special case of depth-zero supercuspidal representations, which are precisely those arising from a datum as above with $n = 0$, except we do not need to assume that G splits over a tamely ramified field extension. The following theorem, a special case of Theorem 4.2.1, is due to Moy and Prasad ([MP94, MP96]) and independently due to Morris ([Mor99]).

Theorem 4.4.1 ([MP94, MP96, Mor99]). *Let $x \in \mathscr{B}(G,F)$ be a vertex. Let (ρ, V_ρ) be an irreducible smooth representation of the stabilizer G_x of x that is trivial on $G_{x,0+}$ and such that $\rho|_{G_{x,0}}$ is a cuspidal representation of the reductive group $G_{x,0}/G_{x,0+}$. Then $\text{c-ind}_{G_x}^{G(F)} \rho$ is a supercuspidal irreducible representation of $G(F)$.*

The above authors also showed that all depth-zero supercuspidal (irreducible smooth) representations are of the form as in Theorem 4.4.1.

Proof of Theorem 4.4.1.
By Lemma 4.3.1 and Fact 4.3.3 it suffices to show that an element $g \in G(F)$ intertwines (ρ, V_ρ) if and only if $g \in G_x$. Since all $g \in G_x$ intertwine (ρ, V_ρ), it remains to show the other direction of the implication. Hence we assume $g \in G(F)$ intertwines (ρ, V_ρ), i.e. we can choose a nontrivial element

$$f \in \mathrm{Hom}_{G_x \cap gG_xg^{-1}}({}^g\sigma, \sigma) \not\simeq \{0\}.$$

Since σ is trivial when restricted to $G_{x,0+}$, the representation ${}^g\sigma$ is trivial when restricted to $gG_{x,0+}g^{-1} = G_{g.x,0+}$. Hence $G_{g.x,0+} \cap G_{x,0}$ acts trivially on the image $\mathrm{Im}(f)$ of f. If $g \notin G_x$, then $g.x \neq x$ and hence by Fact 3.1.6(b) (which also holds for not necessarily split reductive groups), the image of $G_{g.x,0+} \cap G_{x,0}$ in $G_{x,0}/G_{x,0+}$ is the unipotent radical N of a proper parabolic subgroup of $G_{x,0}/G_{x,0+}$. Thus

$$\{0\} \not\simeq \mathrm{Im}(f) \subset V_\rho^N,$$

which contradicts that (ρ, V_ρ) is cuspidal. □

4.5 A parameterization of supercuspidal representations

In Section 4.2 we outlined how to attach supercuspidal representations to a Yu datum, that was described in Section 4.1. In Section 5 we will see that under mild assumptions this provides us with all supercuspidal smooth irreducible representations. In order to parameterize all supercuspidal smooth irreducible representations it therefore remains to understand which Yu data yield the same representation. This has been resolved by Hakim and Murnaghan ([HM08]) up to a hypothesis that was removed by Kaletha ([Kal19, § 3.5]). Hakim and Murnaghan define an equivalence relation on the Yu data, which they call $G(F)$-*equivalence* and the key result is that two supercuspidal representations arising from Yu's construction are equivalent if and only if the input Yu data are $G(F)$-equivalent. In order to define the $G(F)$-equivalence, Hakim and Murnaghan introduced the following three transformations of Yu data.

Definition 4.5.1 (Elementary transformation). A Yu datum $((G_i)_{1\le i\le n+1}, x', (r_i)_{1\le i\le n}, \rho', (\phi_i)_{1\le i\le n})$ is obtained from a Yu datum $((G_i)_{1\le i\le n+1}, x, (r_i)_{1\le i\le n}, \rho, (\phi_i)_{1\le i\le n})$ via an *elementary transformation* if $[x] = [x']$ and $\rho \simeq \rho'$.

Definition 4.5.2 (G-conjugation)**.** We say that a Yu datum is a obtained from the Yu datum $((G_i)_{1\leq i\leq n+1}, x, (r_i)_{1\leq i\leq n}, \rho, (\phi_i)_{1\leq i\leq n})$ via $G(F)$-*conjugation* if it is of the form

$$((gG_ig^{-1})_{1\leq i\leq n+1}, x, (r_i)_{1\leq i\leq n}, {}^g\rho, ({}^g\phi_i)_{1\leq i\leq n})$$

for some $g \in G(F)$.

While the above two operations clearly yield isomorphic representations, there is a third operation on the Yu datum that does not change the isomorphism class of the resulting supercuspidal representation.

Definition 4.5.3 (Refactorization)**.** A Yu datum $((G_i)_{1\leq i\leq n+1}, x, (r_i)_{1\leq i\leq n}, \rho', (\phi'_i)_{1\leq i\leq n})$ is a *refactorization* of a Yu datum $((G_i)_{1\leq i\leq n+1}, x, (r_i)_{1\leq i\leq n}, \rho, (\phi_i)_{1\leq i\leq n})$ if the following two conditions are satisfied.

(i) For $1 \leq i \leq n$, we have

$$\prod_{1\leq j\leq i} \phi_j|_{(G_{i+1})_{x,r_{i+1}+}} = \prod_{1\leq j\leq i} \phi'_j|_{(G_{i+1})_{x,r_{i+1}+}},$$

where we set $r_{n+1} = 0$, and

(ii)

$$\rho \otimes \prod_{1\leq j\leq n} \phi_j|_{(G_{n+1})_{[x]}} = \rho' \otimes \prod_{1\leq j\leq n} \phi'_j|_{(G_{n+1})_{[x]}}.$$

These three operations together allow us to define the desired equivalence relation on the Yu data.

Definition 4.5.4. Two Yu data are $G(F)$-*equivalent* if one can be transformed into the other via a finite sequence of refactorizations, $G(F)$-conjugations and elementary transformations.

The following theorem shows that this is the equivalence relation we were looking for.

Theorem 4.5.5 ([HM08, Kal19])**.** *Two Yu data* $((G_i)_{1\leq i\leq n+1}, x, (r_i)_{1\leq i\leq n}, \rho, (\phi_i)_{1\leq i\leq n})$ *and* $((G'_i)_{1\leq i\leq n+1}, x', (r'_i)_{1\leq i\leq n}, \rho', (\phi'_i)_{1\leq i\leq n})$ *yield isomorphic supercuspidal representations of* $G(F)$ *if and only if they are* $G(F)$*-equivalent.*

For a proof, see [HM08, Theorem 6.6] and [Kal19, Corollary 3.5.5.].

4.6 A twist of Yu's construction

Let $((G_i)_{1\leq i\leq n+1}, x, (r_i)_{1\leq i\leq n}, \rho, (\phi_i)_{1\leq i\leq n})$ be a Yu datum. Instead of associating to this Yu datum the representation $\text{c-ind}_{\widetilde{K}}^{G(F)}\widetilde{\rho}$ constructed by Yu, a new suggestion by Fintzen, Kaletha and Spice ([FKS]) consists of associating the representation $\text{c-ind}_{\widetilde{K}}^{G(F)}(\epsilon\widetilde{\rho})$ for an explicitly constructed character $\epsilon : \widetilde{K} \to \{\pm 1\}$. We refer the reader to [FKS, p. 15] for the definition of ϵ as it is rather involved. There are multiple reasons for the introduction of this quadratic twist in the parametrization. For example, it restores the validity of Yu's original proof ([Yu01]) that $\text{c-ind}_{\widetilde{K}}^{G(F)}(\epsilon\widetilde{\rho})$ is a supercuspidal irreducible representation, which is not valid for the non-twisted version as it relied on a misprinted statement in [Gér77]. In particular, we restore the validity of the intertwining results [Yu01, Proposition 14.1 and Theorem 14.2] for the twisted construction that form the heart of Yu's proof. Instead of stating the results in full generality, which would involve introducing additional notation, we state its implication in the setting that we already introduced above.

Proposition 4.6.1 ([Yu01, FKS]). *Let $((G = G_1 \supsetneq G_2 = G_{n+1}), x, (r_1 = r), \rho, (\phi_1 = \phi))$ be a Yu datum from which we construct a representation κ of $\widetilde{K} = G_{x,\frac{r}{2}}(G_2)_{[x]}$ as in Section 4.2. Then for $g \in G_2(F)$, we have*

$$\dim_{\mathbb{C}} \text{Hom}_{\widetilde{K}\cap g\widetilde{K}g^{-1}}(\epsilon\kappa, {}^{g}(\epsilon\kappa)) = 1.$$

This result also holds in a more general setting in which we drop the assumption that $Z(G_2)/Z(G)$ is anisotropic. We refer the reader to [FKS, Corollary 4.1.11 and Corollary 4.1.12] for the detailed statements and proofs.

Applications of the existence of the above quadratic character $\epsilon : \widetilde{K} \to \{\pm 1\}$ include being able (under some assumptions on F) to provide a character formula for the supercuspidal representations $\text{c-ind}_{\widetilde{K}}^{G(F)}(\epsilon\widetilde{\rho})$ ([Spi18, Spi, FKS]), to suggest a local Langlands correspondence for all supercuspidal Langlands parameters ([Kal]) and to prove the stability and many instances of the endoscopic character identities for the resulting supercuspidal L-packets that such a local Langlands correspondence is predicted to satisfy ([FKS]).

5 Exhaustiveness of the construction of supercuspidal representations

In the previous section we have seen how to construct supercuspidal smooth irreducible representations of a p-adic group $G(F)$. In this section we will see that under some minor assumptions the above construction by Yu provides us with all supercuspidal smooth irreducible representations.

5.1 Exhaustiveness result

Theorem 5.1.1 ([Kim07, Fin21d]). *Suppose that G splits over a tamely ramified field extension of F and that p does not divide the order of the (absolute) Weyl group of G. Then every* *supercuspidal smooth irreducible representation of $G(F)$ arises from Yu's construction, i.e. via Theorem 4.2.1.*

This result was shown by Kim ([Kim07]) in 2007 under the additional assumptions that F has characteristic zero and that p is "very large". Her approach was very different from the recent approach in [Fin21d]. Kim proves statements about a measure one subset of all smooth irreducible representations of $G(F)$ by matching summands of the Plancherel formula for the group and the Lie algebra, while the recent approach in [Fin21d] is more explicit and can be used to recursively exhibit a Yu datum for the construction of the given representation. We will give a sketch of the latter approach. The proof consists of two main steps. The first step is to prove that every supercuspidal smooth irreducible representation of $G(F)$ contains a (maximal) datum as defined in [Fin21d], which we recall below, and which can be viewed as a skeleton of a Yu datum. The second step consists of obtaining a Yu datum from that maximal datum and showing that the representation we started with is isomorphic to the one constructed from this Yu datum.

We assume from now on that G splits over a tamely ramified field extension of F and that p does not divide the order of the (absolute) Weyl group of G.

5.2 The datum as in [Fin21d]

A (maximal) datum as defined in [Fin21d] is at the same time a skeleton for a Yu datum and a much more refined version of the so called *unrefined minimal K-type* introduced by Moy and Prasad ([MP94, MP96]).

Definition 5.2.1. Let $n \in \mathbb{Z}_{\geq 0}$. A *datum* of G of length n is a tuple $(x, (X_i)_{1 \leq i \leq n}, (\rho_0, V_{\rho_0}))$ such that it can be extended to a tuple, called *extended datum*,

$$(x, (r_i)_{1 \leq i \leq n}, (X_i)_{1 \leq i \leq n}, (G_i)_{1 \leq i \leq n+1}, (\rho_0, V_{\rho_0}))$$

where

(a) $G = G_1 \supseteq G_2 \supsetneq G_3 \supsetneq \ldots \supsetneq G_{n+1}$ are twisted Levi subgroups of G,

(b) $x \in \widetilde{\mathscr{B}}(G_{n+1}, F) \subset \widetilde{\mathscr{B}}(G, F)$,

(c) $r_1 > r_2 > \ldots > r_n > 0$ are real numbers,

(d) (ρ_0, V_{ρ_0}) is an irreducible representation of $(G_{n+1}^{\mathrm{der}})_{[x],0}/(G_{n+1}^{\mathrm{der}})_{[x],0+}$, where G_{n+1}^{der} denotes the derived subgroup of G_{n+1},

(e) $X_i \in \mathfrak{g}^*_{x,-r_i} \setminus \mathfrak{g}^*_{x,(-r_i)+}$ for $1 \leq i \leq n$

satisfying the following conditions for all $1 \leq i \leq n$

(i) $X_i \in \mathfrak{g}_i^* := \mathrm{Lie}(G_i)^*(F) \subset \mathfrak{g}^*(F)$,

(ii) X_i is generic of depth $-r_i$ at $x \in \widetilde{\mathscr{B}}(G_i, F)$ as an element of $\mathfrak{g}_i^*$ (under the action of G_i) ,

(iii) $G_{i+1} = \mathrm{Cent}_{G_i}(X_i)$.

Here we use the following definition of "generic", which will imply the genericity conditions required for a Yu datum.

Definition 5.2.2. We say that an element $X \in \mathfrak{g}^*(F)$ is *generic of depth r at $x \in \widetilde{\mathscr{B}}(G, F)$* if the G-orbit of X is closed and if there exists a tamely ramified extension E over F and a split maximal torus $T \subset \mathrm{Cent}_G(X) \times_F E$ such that

- $x \in \widetilde{\mathscr{A}}(T, E) \cap \widetilde{\mathscr{B}}(G, F)$,
- $X \in \mathfrak{g}^*_{x,r}$,
- for every $\alpha \in \Phi(G_E, T_E)$, we have $X(H_\alpha) = 0$ or $\mathrm{val}(X(H_\alpha)) = r$, where $H_\alpha = d\check{\alpha}(1)$, and
- if $X(H_\alpha) = 0$ for all $\alpha \in \Phi(G, T)$, then $X \notin \mathfrak{g}^*_{x,r+}$.

Given a datum, we write H_i for the derived subgroup of G_i for $1 < i \leq n - 1$, and we write $H_1 = G_1$ if $G_1 = G_2$ and otherwise we write H_1 for the derived subgroup of G_1. We choose a maximal torus T of G_{i+1} such that $x \in \widetilde{\mathscr{A}}(T, E)$, where E denotes a finite tamely ramified extension of F of ramification degree e over which T splits. Then we define for a non-negative real number r

$$(H_i)_{x,r} := H_i(F) \cap (G_i)_{x,r} \,, \quad (H_i)_{x,\frac{r}{2}+} := H_i(F) \cap (G_i)_{x,\frac{r}{2}+}$$

and

$$\begin{aligned}(H_i)_{x,r,\frac{r}{2}+} := H(F) \cap \big\langle T(E)_r, U_\alpha(E)_{x,r}, U_\beta(E)_{x,\frac{r}{2}} \quad | \quad & \alpha \in \Phi((G_{i+1})_E, T_E) \subset \Phi(G_E, T_E), \\ & \beta \in \Phi((G_i)_E, T_E) \setminus \Phi((G_{i+1})_E, T_E) \big\rangle \,.\end{aligned}$$

This definition is independent of the choice of T and E ([Yu01, p. 585 and p. 586]). We define the subalgebras $\mathfrak{h}_i$, $(\mathfrak{h}_i)_{x,\frac{r}{2}+}$ and $(\mathfrak{h}_i)_{x,r,\frac{r}{2}+}$ of $\mathfrak{g}$ analogously.

Let (π, V) be a smooth irreducible representation of $G(F)$. We set $r_{n+1} = 0$ and say that a datum $(x, (X_i)_{1 \leq i \leq n}, (\rho_0, V_{\rho_0}))$ of G is *contained in* (π, V) if $V^{\cup_{1 \leq i \leq n+1}((H_i)_{x,r_i+})}$ contains a subspace V' such that

- $(\pi|_{(H_{n+1})_{x,0}}, V')$ is isomorphic to (ρ_0, V_{ρ_0}) as a representation of $(H_{n+1})_{x,0}/(H_{n+1})_{x,0+}$ and
- $(H_i)_{x,r_i,\frac{r_i}{2}+}/(H_i)_{x,r_i+} \simeq (\mathfrak{h}_i)_{x,r_i,\frac{r_i}{2}+}/(\mathfrak{h}_i)_{x,r_i+}$ acts on V' via the character $\varphi \circ X_i$ for $1 \leq i \leq n$,

where $\varphi : F \to \mathbb{C}^*$ is the additive character of F that is trivial on $\varpi\mathcal{O}$ that we fixed above.

We caution the reader that at this stage the representation (ρ_0, V_{ρ_0}) is more like a place holder and is not the same as the representation ρ that forms part of a Yu datum. In fact, retrieving ρ from ρ_0 is a key task in the second step of the proof that all supercuspidal representations arise form Yu's construction. In the first step the main focus is on constructing the generic elements X_i recursively.

Theorem 5.2.3 ([Fin21d])**.** *Let (π, V) be a smooth irreducible representation of $G(F)$, and recall that we assume that G splits over a tamely ramified field extension and p does not divide the order of the (absolute) Weyl group of G. Then (π, V) contains a datum.*

Note that we do not assume that (π, V) is supercuspidal. This result works without the assumption of supercuspidality and will lead to the notion of types in general, which we will not elaborate in this survey. As mentioned above, the proof proceeds recursively, i.e. by first showing the existence of a suitable element X_1 so that the action of $(H_1)_{x,r_1,\frac{r_1}{2}+}/(H_1)_{x,r_1+}$ is given by $\varphi \circ X_1$. Let us denote the resulting character of $(H_1)_{x,r_1,\frac{r_1}{2}+}$ by χ_1 and assume G is semisimple for now for simplicity, which implies $H_1 = G_1 = G$. Then $(G_{x,r_1}, \chi_1|_{G_{x,r_1}})$ is an unrefined minimal K-type of depth r_1 contained in (π, V) as introduced by Moy and Prasad. However, the genericity condition imposed on X_1 described above is slightly stronger than the non-degeneracy condition that Moy and Prasad imposed on a pair $(G_{x,r_1}, \chi_1|_{G_{x,r_1}})$ to be called an unrefined minimal K-type. The slightly stronger condition is crucial for the recursion steps to work and to eventually recover a Yu datum. After proving the existence of a suitable element X_1, the proof consists of recursively finding $X_2, X_3, \ldots$, but additional difficulties arise when constructing X_2 from ensuring that not only should the action of $(H_2)_{x',r_2,\frac{r_2}{2}+}$ be given by X_2, but also the action of $(H_1)_{x',r_1,\frac{r_1}{2}+}$ should still be given by X_1. As our notation indicates, the initial point $x \in \widetilde{\mathscr{B}}(G, F)$ used to find X_1 might not be suitable and has to be changed to a different point x' to achieve this result. The required arguments to show that the recursion works form a significant part of [Fin21d] and we refer the reader to [Fin21d, Proof of Theorem 6.1] for the details.

5.3 Recovering a Yu datum from a supercuspidal representation

Let (π, V) be a smooth irreducible supercuspidal representation of $G(F)$. The second step in the proof of Theorem 5.1.1 consists of refining a datum contained in (π, V) to obtain a Yu datum from which (π, V) can be constructed. To this end, we start with a maximal datum rather than an arbitrary datum. We call a datum $(x, (X_i)_{1\leq i\leq n}, (\rho_0, V_{\rho_0}))$ contained in (π, V) a *maximal datum* for (π, V) if given another datum $(x', (X_i)_{1\leq i\leq n}, (\rho_0', V_{\rho_0}'))$ contained in (π, V), we have that the dimension of the facet of $\widetilde{\mathscr{B}}(G_{n+1}, F)$ that contains x is at least the dimension of the facet of $\widetilde{\mathscr{B}}(G_{n+1}, F)$ that contains x'. A maximal datum provides a twisted tame Levi sequence $(G_i)_{1\leq i\leq n+1}$, a point $x \in \widetilde{\mathscr{B}}(G_{n+1}, F)$ and real numbers $r_1 > r_2 > \ldots > r_n > 0$, and it remains to find an appropriate irreducible representation ρ of $(G_{n+1})_{[x]}$ and characters ϕ_i of $G_{i+1}(F)$ that together form a Yu datum and whose associated representation is isomorphic to (π, V). The characters ϕ_i are constructed recursively from the elements X_i and their genericity properties result from the X_i being generic. The construction of ρ and the proof that ρ is cuspidal uses the theory of Weil–Heisenberg representations together with the property of the datum being a maximal datum for (π, V). We refer the reader to [Fin21d, § 7] for the details. So far we have not used that (π, V) is supercuspidal, and indeed, for readers who know about types, we remark that we obtain for every smooth irreducible representation of $G(F)$ an input for the construction of a type by Kim and Yu ([KY17]) for the corresponding Bernstein block. When (π, V) is supercuspidal we can prove that the remaining conditions for a Yu datum are satisfied ([Fin21d, § 8]) and through the way that the Yu datum is obtained we ensure that the resulting representation $\widetilde{\rho}$ of $\widetilde{K}$ constructed by Yu is contained in $(\pi|_{\widetilde{K}}, V)$. Using Frobenius reciprocity we deduce that (π, V) is isomorphic to $\text{c-ind}_{\widetilde{K}}^{G(F)} \widetilde{\rho}$.

Selected notation

Selected definitions

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