Eighth exercise sheet "Algebra II" winter term 2024/5.

Problem 1 (4 points)**.** *Let A be a discrete valuation ring with uniformizing element π and valuation v. Let $P(T) = T^d + \sum_{j=0}^{d-1} p_j T^j \in A[T]$ be a polynomial such that $v(p_j) > 0$ for $0 \le j < d$ and $v(p_0) = 1$. Show that $S = A[T]/PA[T]$ is a discrete valuation ring with maximal ideal TS!*

Problem 2 (2 points)**.** *In the situation of the previous problem, calculate the prime ideal decomposition of πS!*

From now on let p be a prime number and K be an algebraic number field such that $\mathcal{O}_K$ is unramified over $\mathbb{Z}$ at all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ containing p. Let $q = p^k$ for some positive integer k, and let $q' = p^{k-1}$. Let

$$P_q(T) = \frac{T^q - 1}{T^{q'} - 1} = \sum_{j=0}^{p-1} T^{jq'}$$

be the q-th cyclotomic polynomial.

Problem 3 (5 points)**.** *Let $A = \left(\mathcal{O}_K\right)_{\mathfrak{p}}$ where $\mathfrak{p} \in \operatorname{Spec}\mathcal{O}_K$ contains $\mathfrak{p}$. Show that Problem 7 from the previous sheet and the current Problem 1 can be applied to A and the polynomial $P(T) = P_q(T+1)$!*

From now on we put $L = K(\zeta) = K(\mu_q)$, where ζ is a generator of the cyclic group $\mu_q \subseteq \mathbb{C}$. We also change notations from the previous problem and put $A = \mathcal{O}_K$. We also put $B = A[\zeta]$.

Problem 4 (2 points)**.** *Deduce from the result of the previous problem that $B_{\mathfrak{p}} = \left(\mathcal{O}_L\right)_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}A$ containing p!*

Problem 5 (2 points)**.** *Show that q is a multiple of $P_q'(\zeta)$ (the derivative of P_q at ζ) in B!*

Problem 6 (3 points)**.** *Show that $B_p = \left(\mathcal{O}_L\right)_p$!*

Problem 7 (6 points)**.** *Conclude that $A = B$! Moreover, for every $\mathfrak{p}$ as in Problem 4 determine the number of prime ideals of B above $\mathfrak{p}$, the degree of their residue field extension and their exponent in the prime ideal decomposition of $\mathfrak{p}B$! Moreover, depending on q decide whether B/A is unramified, tamely ramified or wildly ramified over $\mathfrak{p}$!*

Four of the 24 points available from this exercise sheet are bonus points which are disregarded in the calculation of the 50%-limit for passing the exercises.

Solutions should be submitted to the tutor by e-mail before Friday December 6 24:00.