Bertrand's Conjecture: At least one Prime between n and $2n$ *

In 1845 Joseph Bertrand published the conjecture that for any integer n there is always at least one prime $p \in (n, 2n)$ – after checking this numerically up to $n = 3\,000\,000$.

In 1850 P. Tschebyschew had discovered how much information about primes can be derived by studying the binomial coefficients. He published a proof of Bertrand's conjecture and a preliminary version of the prime number theorem, namely

$$0.92929\frac{x}{\log x} \leq \pi(x) \leq 1.1056\frac{x}{\log x}, \text{ with } \pi(x) := \textit{number of primes } \leq x.$$

Both proofs are based on looking at the prime factors of binomial coefficients.

I hope to explain the proof of the Bertrand conjecture that I learnt from Aigner-Ziegler in such a way that a first term mathematics student might get an idea how a genius could have found such a proof.

First one has to find some facts which seem connected to the primes $p \in (n, 2n)$. A century before Tschebyschew the French mathematician Legendre had looked at the prime factors of binomial coefficients. He made this observation:

(P1) **All** primes $p \in (n, 2n)$ are clearly simple prime factors of $\binom{2n}{n} = \frac{(2n)!}{n!n!}$, because they divide the numerator once and do not divide the denominator. Moreover, no larger primes divide $(2n)!$, hence of course not $\binom{2n}{n}$.

Trivially, a number is equal to the product of all its prime factors (to the right powers). The assumption, that Bertrand's conjecture is false for some number n, therefore has the drastic consequence that $\binom{2n}{n}$ has to be a product only of primes $p \in [2, n]$. This lets us see a glimpse of a possible strategy: If the product of (the correct powers of) the smaller primes, $p \in [2, n]$, cannot be as large as $\binom{2n}{n}$, then we have a contradiction that proves the Bertrand conjecture.

The product formula for the binomial coefficients does not give a good idea how large these coefficients may be. However, the largest of the coefficients $\binom{n}{k}$, $k = 0, 1, \ldots, n$, is clearly smaller than their sum $\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n$. Hence

(B1) $\binom{n}{k} \leq 2^n, \quad \binom{2n}{n} \leq 4^n$.

Did this easy proof give a good bound? The largest binomial coefficient of course is also larger than the average of the n numbers $\binom{n}{0} + \binom{n}{n}, \ \binom{n}{1}, \ldots, \binom{n}{n-1}$. Their sum is 2^n, hence

(B2) $\max_k \binom{n}{k} \geq 2^n/n, \quad \binom{2n}{n} \geq 4^n/2n$.

Comment: Since $2n = e^{\log(2n)} < 4^{\log(2n)}$ we have $4^{n-\log(2n)} < \binom{2n}{n} \leq 4^n$. This says that the exponents of the upper and lower bounds are really close together since $1000 < 2^{10}$ implies $\log(1000) < 10, \ \log(1000\,000) < 20, \ \log(10^{3k}) < 10k$. Or: $\log(2n)$ is only a small fraction of n (the larger n, the smaller the fraction). – (B1) and (B2) look like good information.

---

* This text follows Aigner-Ziegler's presentation of Erdös' arguments in PROOFS FROM THE BOOK. I tried to rewrite it for mathematically less educated readers.

A slightly more precise glimpse of a possible strategy becomes visible: If the product of (the correct powers of) the smaller prime factors of $\binom{2n}{n}$, $p \in [2, n]$, could be proved to be smaller than $4^n/2n$ then there have to be prime factors in $(n, 2n)$, i.e. the Bertrand conjecture would follow.

At this point one would look up what Legendre knew about the small prime factors of $\binom{2n}{n}$. By asking Sokrates type questions, one can also figure it out. First, if n were itsself prime, $n = p$, then p^2 would divide $(2n)!$ and also $n! \cdot n!$, but p^3 would not. Therefore this p would **not** divide $\binom{2n}{n}$. For which primes $p \le n$ does this argument work? We only used $2n < 3p$. Therefore we have further support for the envisioned strategy:

(P2) **No** prime $p \in (\frac{2}{3}n, n]$ is a factor of $\binom{2n}{n}$.

Next we ask: To what power does a prime p divide $m!$ and are there assumptions that make the answer easier? Imagine all the multiples $p, 2p, \ldots, lp$ in the sequence $1, 2, \ldots, m$. If $m < p^2$ then p^l, $l := [m/p]$(:= *the largest integer* $\le m/p$) is the **exact** power of p that divides $m!$. We conclude (and find more support for the strategy):

(P3) The primes $p \in (\sqrt{2n}, \frac{2}{3}n]$ divide $\binom{2n}{n}$ exactly $[2n/p] - 2[n/p]$ times. And: $[2n/p] - 2[n/p]$ **<** $2n/p - 2(n/p - 1) = 2$. These primes divide $\binom{2n}{n}$ **at most once!**

Next question: Can we find the exact prime powers p^l in $m!$ even if $p^2 \le m$ but $m < p^3$? The answer is simple: Each time we find among the multiples of p in $1, 2, \ldots, m$ a multiple of p^2 we get **one more** factor p than was already counted by looking at the multiples of p: we have to count how many multiples of p plus those of p^2 we find. The exact power p^l in $m!$ is given by $l := [m/p] + [m/p^2]$. This gives for the binomial coefficient: If $p^2 \le 2n < p^3$ then the exact power of p in $\binom{2n}{n}$ is p^l, $l := ([2n/p] - 2[n/p]) + ([2n/p^2] - 2[n/p^2]) \le 1 + 1$. Clearly, this counting can be continued: For any "small" prime $p \le \sqrt{2n}$ first find the number k such that $p^k \le 2n < p^{k+1}$. The same counting as before gives:

(P4) The precise power of p in $\binom{2n}{n}$ is p^l with $l := \sum_{j=1}^{k}([2n/p^j] - 2[n/p^j]) \le k$, hence $p^l \le p^k \le 2n$ (recall $< p^{k+1}$).

All this information from Legendre is promising, but how can we proceed from here? It looks as if most of the primes we have to deal with are those from (P3). Without further information we need to assume that all of them are factors of the binomial coefficient. The simplest way to proceed would therefore be to find a good upper bound for the product of all primes $p \le x$. In other words, can we find a function $f(x)$ such that $(\prod_{p \le x} p) \le f(x)$? Since such a bound for an odd number x will also work for the next even number $x+1 = 2m$, we can assume by induction that

$$\text{we found } f(x) \text{ for all } x \le 2m \text{ and we need to find } f(2m+1).$$

How can we split the product $(\prod_{p \le 2m+1} p)$ so that we can use $f(x)$ for $x \le 2m$? The proof of (P1) also gives: **All** the primes $p \in (m + 1, 2m + 1]$ are simple prime factors of $\binom{2m+1}{m+1}$ (they divide the numerator once and do not divide the denominator).

Therefore we have

$$\Big(\prod_{m+1<p\le 2m+1} p\Big) \le \binom{2m+1}{m+1}. \text{ And this is } \le 4^m,$$

because the proof of (B1) can be slightly changed: There are **two** largest coefficients

$$\binom{2m+1}{m+1} = \binom{2m+1}{m}, \text{ thus we get: } \binom{2m+1}{m+1} \le \frac{1}{2}(1+1)^{2m+1} = 4^m.$$

Putting these observations together proves:

$$\begin{aligned}\Big(\prod_{p\le 2m+1} p\Big) &= \Big(\prod_{p\le m+1} p\Big)\cdot\Big(\prod_{m+1<p\le 2m+1} p\Big) \le \Big(\prod_{p\le m+1} p\Big)\cdot\binom{2m+1}{m+1} \le \Big(\prod_{p\le m+1} p\Big)\cdot 4^m \\ &\le f(m+1)\cdot 4^m.\end{aligned}$$

This is indeed an induction step because it estimates $(\prod_{p\le 2m+1} p)$ in terms of the **smaller** product $(\prod_{p\le m+1} p)$. The step works for any bounding function f that satisfies

$$f(m+1)\cdot 4^m \;\le\; f(2m+1),$$

namely if $\displaystyle\prod_{p\le m+1} p \le f(m+1)$ then $\displaystyle\prod_{p\le 2m+1} p \le f(2m+1)$.

Such a function is easy to guess, $f(x) = c\cdot 4^x$ will do. The induction can start at $x = 2$ if we put $c = 1/8$ since $2 \le 4^2/8$. Therefore we proved for all $x \ge 2$

$$\prod_{p\le x} p \le \frac{4^x}{8}.$$

We have found enough pieces and can put them together. The decisive question is: Will the result be small enough? The product of the primes that go into $\binom{2n}{n}$ at most to the first power (i.e. the $p \in (\sqrt{2n}, \frac{2}{3}n]$ by (P3)) we estimate by $f(\frac{2}{3}n)$, as just derived. For the primes $p \le \sqrt{2n}$ we use Legendre's bound: their largest power that divides $\binom{2n}{n}$ is $p^l \le 2n$. Except for initial crowding there are less primes than odd numbers. If $\sqrt{2n} \ge 14$ then below $\sqrt{2n}$ are at most $\sqrt{2n}/2 - 1$ primes.

The following **principal estimate** bounds the product of the prime powers in $\binom{2n}{n}$, under the assumption that the Bertrand conjecture is false, as follows:

$$4^n/2n \;\le\; \binom{2n}{n} \;\le\; (2n)^{\sqrt{2n}/2-1}\cdot\Big(\prod_{\sqrt{2n}<p\;\le\;\frac{2}{3}n} p\Big) \;\le\; (2n)^{\sqrt{2n}/2-1}\cdot 4^{\frac{2}{3}n}/8,$$

or

$$\underline{\underline{8\cdot 4^{2n/6} \;\le\; (2n)^{\sqrt{2n}/2}.}}$$

But why is this a contradiction? With the inequality $4^5 = 1024 > 10^3$ we can elementarily show the **opposite** for even powers of 10, $2n = 10^{2k}$, by using $10^k \geq 10k$ which is true (by induction) for integers $k \geq 1$. Namely:

$$8 \cdot 4^{2n/6} = 8 \cdot 4^{10^{2k}/6} > 10^{10^{2k}/10} \geq 10^{k10^k} = (10^{2k})^{10^k/2} = (2n)^{\sqrt{2n}/2}.$$

With a small amount of analysis one can prove the inequality $10^r \geq 10r$ for all **real** $r \geq 1$ (show: the derivative of $10^x - 10x$ is positive in $[1, \infty)$). This establishes the contradiction for $2n \geq 14^2$. It remains to check the conjecture numerically up to $n = 98$. The following primes $\{2, 3, 5, 7, 13, 23, 43, 83, 163\}$, which satisfy $p_{i+1} < 2p_i$, provide this check to $n = 162$.

[A break of style? If one does not want to quote any analysis at all, one can use the inequality $10^k \geq 100k + 20$ (true by induction for $k \geq 3$) to prove

$$4 \cdot 4^{2n/6} > (100 \cdot 2n)^{\sqrt{100 \cdot 2n}/2} \quad \text{for} \quad 2n = 10^{2k}, \quad k \geq 3, \qquad \text{hence:}$$
$$4 \cdot 4^{2n/6} > (2n)^{\sqrt{2n}/2} \quad \text{for} \quad 2n = 10^{2k}, 10^{2k} + 2, 10^{2k} + 4, \ldots, 10^{2(k+1)}, k \geq 3.$$

In that case it remains to check the Bertrand conjecture numericaly up to $2n = 10^6$.]

Final remark. The crucial part of the principal inequality is that the exponent $2n/6$ on the left grows **faster** than the exponent $\sqrt{2n}/2$ on the right. This allows to generalize the conjecture for example to:

In every interval $(n, 2n)$ are more than $\sqrt{2n}/2$ primes.

The same indirect argument as above gets $\sqrt{2n}/2$ additional factors of primes $p \in (n, 2n)$ into the upper bound estimate (underlined below), giving

$$4^n/2n \;\leq\; \binom{2n}{n} \;\leq\; (2n)^{\sqrt{2n}/2-1} \cdot \Big(\prod_{\sqrt{2n} < p \leq \frac{2}{3}n} p \Big) \cdot \underline{\underline{(2n)^{\sqrt{2n}/2}}} \;\leq\; (2n)^{\sqrt{2n}-1} \cdot 4^{\frac{2}{3}n}/8,$$

or

$$8 \cdot 4^{2n/6} \;\leq\; (2n)^{\sqrt{2n}}.$$

This is wrong for $2n \geq 900$, the new conjecture can be checked numerically for $2n < 900$.