English Summary of

# Argumente der Analysis

The text contains the motivations and the proofs for an Analysis course with a novel arrangement of the material. But it omits some computations that first year students probably should see.

It does **not** follow the standard route:
- sequences and series,
- continuity,
- differentiability,
- properties of limit functions.

Rather, it starts with the differentiation of polynomials, and proves our main tool, the **monotonicity theorem**, before dealing with the completeness of the reals:

**Monoticity Theorem.** *If a polynomial $P$ has $P' \geq 0$ on some interval, then $P$ is weakly increasing on that interval.*

This is possible for the following reason: The formula $x^n - a^n = (x-a)\cdot(x^{n-1}+\ldots+a^{n-1})$ can be used twice to prove for the polynomial $P(x) := \sum_0^n a_k x^k$ on the intervall $[-R, R]$ with the constant $K := \sum_2^n |a_k| k(k-1) R^{k-2}$ the bound for the deviation from the tangent:

$$|P(x) - P(a) - P'(a)(x-a)| \leq K \cdot |x-a|^2$$

This estimate is good enough to prove the monotonicity theorem if it is combined with:

**Archimedes' Principle.** *If an inequality $a \leq b + 1/n$ holds for every $n \in \mathbb{N}$ then $a \leq b$ holds.*

The monotonicity theorem has (because of the quotient rule) a multiplicative version:

**Monoticity Theorem (Multiplicative version).** *If positive functions $f$ and $g$ satisfy the growth rate assumption $f'/f < g'/g$, then $g/f$ is an increasing function.*

These tools are powerful and allow us to construct approximations of wanted new functions like $\exp, \sin, \cos$. To get the functions themselves we only need limits and **not** the usual interplay between continuity and completeness plus uniform convergence. The monotonicity theorem gives Lipschitz bounds for the approximations $f_n$ of new functions and since these bounds are uniform, i.e. indepndent of $n$, Archimedes Principle gives the same Lipschitz bound for the limit function.

Why would one want such an approach? The development of polynomials as a basic tool is more important for most future users of mathematics than spending a long period on sequences and continuity before introducing them to differentiation. Moreover the explicit estimates are conceptually simpler than the infinite set of $\epsilon$-$\delta$-implications needed to describe continuity.

Next, complex numbers are introduced and the previous developments are repeated in this context. We treat complex power series both to develop the elementary functions further, and to practice majorization by geometric series—another fundamental tool in analysis.

Integrals of functions which have (as all our examples do) uniform approximations by piecewise linear functions are treated easily; we try to get the interpretation of an integral as a *continuous sum* across. Finally, continuous functions appear as those which map convergent sequences to convergent sequences. With all the previous experience available the proofs of the standard theorems about continuous functions are now much simpler than at the beginning of the theory.

The next section (not yet finished) uses ordinary differential equations (ODEs) to illustrate how the qualitative concept of continuity is much easier to use in this context than our earlier quantitative methods, but the latter are needed again to control numerical methods.