Free Rotational Motion of Rigid Bodies

Hermann Karcher

Part I: Angular Velocity and Rigid Motion

In this first part we will not yet consider solid objects with their inertial properties, but only so-called *rigid body kinematics*, i.e., the study of rotational motions of space. The (simpler) particle mechanics analogue of the question that we will discuss is the following: knowing the velocity curve $v(t)$ of a point how can we reconstruct the travel path $c(t)$? Since $c'(t) = v(t)$, $c(t)$ is an antiderivative of $v(t)$ and we can find it easily by integration. (Historically $v(t)$ was called the *hodograph* of the motion.)

Things to try in 3D-XplorMath

The last three entries of the Action Menu of Space Curves show demos that illustrate the present discussion. The first of these Actions, *Use Curve as Hodograph*, interpretes the space curves of 3D-XplorMath as velocity curves of a particle and reconstructs the path. The demo emphasizes that the tangent vector of the constructed path is (parallel to) the position vector of the selected space curve, the hodograph.

The second of these Actions, *Use Curve as Angular Velocity* $\vec{\omega}(t)$, reconstructs the rotational motion which has the given space curve as given angular velocity function. The visualization of the motion uses a sphere with random dots and shows several consecutive points of the orbit of each random dot. One sees large orbit velocities near the equator of the rotation and small velocities near the axis of the rotation *at each moment.* – More details are explained below.

The third of these Actions, *Use Curve as Components of* $\vec{\omega}(t)$ *in the Moving System*, again reconstructs that rotational motion that has its angular velocity given in the moving system by the selected space curve. The space curve therefore rotates with the motion. It leaves a trace behind which shows the corresponding angular velocity curve in the observer's space. In the second Action this curve was the given one.

Finally, there is one very special space curve, *Solid Body (Euler's Polhode).* If this space curve is selected for the third Action above then the resulting motion is *the physical motion* around the center of mass of a rigid body, taken to be a brick with edge lengths $aa \geq bb \geq cc$ and initial components of the angular momentum dd, ee, ff, see the ATO of Solid Body.

Angular Velocity given in the Observer's Space

Mathematicians and Physicists have slightly different pictures of a *motion* in their minds. A physicist sees a solid object moving in space, the movement is differentiable and all points $\vec{x}_i(t)$ of the moving object have their orbit velocities $\vec{x}_i{}'(t)$. So far these functions could also describe a mass of moving air. The word *rigid motion* means that the pairwise distances $|\vec{x}_i(t) - \vec{x}_j(t)|$ remain *constant* in time – the points $\vec{x}_i(t)$ could be the atoms of a stone. For a mathematician on the other hand the primary concept is that of a *distance preserving map of space*, and a motion is a 1-parameter family of such maps. For physicists and mathematicians it is important to understand the velocity fields $\vec{x}_i{}'(t)$ of all the particles. Physicists begin by studying rotations around *fixed* axes with *constant* angular velocities. In such a situation one can compute all the velocities $\vec{x}_i{}'(t)$ from *one* vector $\vec{\omega}$ that is parallel to the rotation axis and from the particle positions $\vec{x}_i(t)$ as follows:

$$\vec{x}_i{}'(t) = \vec{\omega} \times \vec{x}_i(t).$$

It is now a mathematical fact that differentiable families of distance preserving maps have a very similar formula for the velocities of the particles: For each time t there exists a vector $\vec{\omega}(t)$ such that we have:

$$\vec{x}_i{}'(t) = \vec{\omega}(t) \times \vec{x}_i(t).$$

And vice versa, if such a relation between the velocities and the positions holds then all pairwise distances between the particles are constant in time. Therefore mathematicians and physicists agree that a differentiable rigid motion is characterized by this relation between particle positions and particle velocities.

Now, a natural question is: If $\vec{\omega}(t)$ is a given vector function in $\mathbb{R}^3$, how can one reconstruct the rotational motion? We answer this question by constructing a so called *moving frame* $\{\vec{e}_x(t),\ \vec{e}_y(t),\ \vec{e}_z(t)\}$, a time dependent orthonormal basis. To do this we have to solve the following three ODEs:

$$\begin{aligned}
\vec{e}_x{}'(t) &= \vec{\omega}(t) \times \vec{e}_x(t), \quad \vec{e}_x(0) = (1,0,0) \\
\vec{e}_y{}'(t) &= \vec{\omega}(t) \times \vec{e}_y(t), \quad \vec{e}_y(0) = (0,1,0) \\
\vec{e}_z{}'(t) &= \vec{\omega}(t) \times \vec{e}_z(t), \quad \vec{e}_z(0) = (0,0,1).
\end{aligned}$$

Next we observe that *all* linear combinations with *constant* coefficients, i.e.

$\vec{x}(t) := x \cdot \vec{e}_x(t) + y \cdot \vec{e}_y(t) + z \cdot \vec{e}_z(t)$ satisfy $\vec{x}\,'(t) = \vec{\omega}(t) \times \vec{x}(t)$ and are therefore orbits of the rotational motion defined by the angular velocity $\vec{\omega}(t)$. To visualize this motion observe that for each *fixed* t the velocity field $\vec{v}(\vec{x}) := \vec{\omega}(t) \times \vec{x}$ **is** the velocity field of the ordinary rotation around the axis $\vec{\omega}(t)\mathbb{R}$ with constant angular velocity $|\vec{\omega}(t)|$.

Angular Velocity given in the Moving Space

What could it mean *to give the angular velocity of a motion in moving space*? We saw in the previous discussion that we can describe the motion of space by giving a *moving frame* $\{\vec{e}_x(t),\ \vec{e}_y(t),\ \vec{e}_z(t)\}$. The particles of moving objects have position vectors that have *constant* components a_x, a_y, a_z relative to this frame: $\vec{x}_i(t) = a_x\vec{e}_x(t) + a_y\vec{e}_y(t) + a_z\vec{e}_z(t)$. Similarly we can prescribe $\vec{\omega}(t)$ by giving its components relative to the moving frame: $\{\omega_x(t), \omega_y(t), \omega_z(t)\}$. There is again a natural question: can we again reconstruct a corresponding rotational motion for any vector function $\vec{\omega}(t)$ that is given in this way?

The answer is almost the same as for the first question, except that the three ODEs are no longer separate but are coupled by the fifth line:

$$\vec{e}_x{}'(t) = \vec{\omega}(t) \times \vec{e}_x(t), \quad \vec{e}_x(0) = (1, 0, 0)$$
$$\vec{e}_y{}'(t) = \vec{\omega}(t) \times \vec{e}_y(t), \quad \vec{e}_y(0) = (0, 1, 0)$$
$$\vec{e}_z{}'(t) = \vec{\omega}(t) \times \vec{e}_z(t), \quad \vec{e}_z(0) = (0, 0, 1)$$

with

$$\vec{\omega}(t) = \omega_x(t) \cdot \vec{e}_x(t) + \omega_y(t) \cdot \vec{e}_y(t) + \omega_z(t) \cdot \vec{e}_z(t).$$

Historical note: The given curve $\{\omega_x(t), \omega_y(t), \omega_z(t)\}$ in the moving system is called the *polhode* of the motion and the corresponding curve $\vec{\omega}(t) = \omega_x(t) \cdot \vec{e}_x(t) + \omega_y(t) \cdot \vec{e}_y(t) + \omega_z(t) \cdot \vec{e}_z(t)$ in the inertial space is called the *herpolhode*. The moving polhode and the fixed herpolhode touch each other at each time t with tangents of equal length – because the points on the momentary axis of rotation, $\vec{\omega}(t)\mathbb{R}$, have at time t the rotational velocity field $\vec{\omega}(t) \times \vec{\omega}(t) = \vec{0}$ in $\mathbb{R}^3$. A visual interpretation of this fact is that the moving polhode rolls without slipping along the fixed herpolhode. (This description actually determines the rotational motion because the origin is fixed so that the polhode has no freedom to rotate around the common tangent with the herpolhode, there is only one way to roll along.)