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Introduction to the Complex Analysis of Minimal Surfaces

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Books: Osserman,R., Nitsche,J.C.C., Dierkes-Hildebrandt-Küster-Wohlrab,
Hoffman-Karcher in Encyclopaedia of Math. Sciences, Vol 90, Geometry V.

Let $c : I \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{R}$. To define the differential $df_{|p} : \mathbb{C} \to \mathbb{R}$ of the function f choose a tangent vector $v \in \mathbb{C}$ at p, pick a curve c such that $c(0) = p,\ c'(0) = v$ and set:

$$df_{|p} \cdot v := df_{|p}(v) := (f \circ c)'(0).$$

The differentials $dx_{|p},\ dy_{|p}$ of the functions $x(z) := \mathrm{Re}\,(z)$ and $y(z) := \mathrm{Im}\,(z)$ are not depending on the base point p. They are always denoted dx, dy and they are taken as basis for representing other differentials or differential forms (= linear combinations of differentials, functions as coefficients).
In $\mathbb{C}$ one uses always the standard basis $e_x = \mathbf{1},\ e_y = \mathbf{i}$ and therefore writes:

$$v = v_x \cdot \mathbf{1} + v_y \cdot \mathbf{i},\ \ df_{|p}(v) = v_x df_{|p}(\mathbf{1}) + v_y df_{|p}(\mathbf{i}) = v_x \frac{\partial}{\partial x} f_{|p} + v_y \frac{\partial}{\partial y} f_{|p},$$

$$df = \frac{\partial}{\partial x} f \cdot dx + \frac{\partial}{\partial y} f \cdot dy, \qquad \text{(differential)}$$

$$\omega = \omega(\mathbf{1}) \cdot dx + \omega(\mathbf{i}) \cdot dy \qquad \text{(differential form)}.$$

A function $F : D \subset \mathbb{C} \to \mathbb{C}$, differentiable in the real sense, is called *complex differentiable* or *holomorphic* if and only if (= iff) for all $v \in \mathbb{C}$ one has

$$dF(\mathbf{i}v) = \mathbf{i}dF(v),$$

or, using the decomposition $F = f + i \cdot g,$ one obtains the Cauchy-Riemann equations:

$$dg(v) = -df(\mathbf{i}v), \qquad \text{or} \qquad \frac{\partial}{\partial x} f = \frac{\partial}{\partial y} g,\ \frac{\partial}{\partial y} f = -\frac{\partial}{\partial x} g.$$

This says among other things that the imaginary part g of F can be obtained from the real part f by integrating the differential form $\omega := -df(\mathbf{i}\cdot\ ) = (-\frac{\partial}{\partial y} f \cdot dx + \frac{\partial}{\partial x} f \cdot dy)$. Namely, choose a curve c which joins a fixed base point $z_* = c(0)$ to $z = c(1)$ then we have

$$g(z) - g(z_*) = \int_0^1 (g \circ c)'(t) dt = \int_0^1 dg_{|c(t)}(c'(t)) dt$$

$$= \int_0^1 -df_{|c(t)}(\mathbf{i}c'(t)) dt = \int_0^1 \omega_{|c(t)}(c'(t)) dt.$$

Note that this also says how one would have to define the integral of a differential form if one did not know already.

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We generalize these concepts to surfaces. Let $\Psi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be an immersed surface together with a chosen unit normal field $N : D \subset \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$ with $|N| = 1$ and $N(p) \perp \text{image}(d\Psi_{|}p)$ (called the spherical Gauss map). We define on each tangent space of the surface a 'positive' 90° rotation by sending a tangent vector X of the surface to $\text{Rot}(90°)(X) := N \times X$. When we differentiate we will usually work in the domain of Ψ and therefore extend the definition. We denote the 90° rotation in the domain D by$J$:

$$X_{|\Psi(p)} := d\Psi_{|p}(v), \qquad d\Psi_{|p}(J \cdot v) := N(p) \times d\Psi_{|p}(v).$$

Next we define when a differentiable function from the surface to $\mathbb{C}$ is called *holomorphic*. To prepare us for doing the same on manifolds we use the identification between D and the surface by the parametrization Ψ and write the function as $F : D \to \mathbb{C}$ (so that $F \circ \Psi^{-1}$ is really the map from the surface to $\mathbb{C}$ and Ψ^{-1} is a local coordinate map for the surface).

A differentiable map $F : D \to \mathbb{C}$ is called *holomorphic* iff $dF(J \cdot v) = \mathbf{i} \cdot dF(v)$. ||

Such F are conformal maps. Decomposition into real and imaginary part $F = f + \mathbf{i} \cdot g$ gives $dg(v) = -df(J \cdot v)$, still called Cauchy-Riemann equations. Again, given a real function f, it is the real part of a conformal (hence holomorphic) map $f + \mathbf{i} \cdot g$ iff the differential form $\omega := -df(J\cdot\ )$ is the differential of a function.

There is a good criterion for a differential 1-form ω to be the differential of a function: its exterior derivative must vanish, $d\omega = 0$. In the basis representation $\omega = \omega_x \cdot dx + \omega_y \cdot dy$ one has the formula

$$d\omega = (\frac{\partial}{\partial x}\omega_y - \frac{\partial}{\partial y}\omega_x) \cdot dx \wedge dy$$

Applied to the differential form above, $\omega := -df(\mathbf{i}\cdot\ ) = (-\frac{\partial}{\partial y}f \cdot dx + \frac{\partial}{\partial x}f \cdot dy)$, this criterion gives the well known fact $\Delta f := (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})f = 0$, namely that the harmonic functions are the real parts of the holomorphic functions — on simply connected domains at least.

This local computation we cannot directly generalize to answer the question which real functions on a surface are the real parts of conformal maps (we would have to use the *covariant derivative* of the Riemannian metric given by the first fundamental form of the surface). This can be avoided since many differential forms in Differential Geometry come up without the need to express them with respect to a local basis. To use this we need another expression for the exterior derivative.
We use $D \subset \mathbb{R}^2$ to take the difference of vectors at different points, hence to compute the directional derivative of a vectorfield X on D as $\frac{d}{dt}X(c(t))_{|t=0} =: T_{c'(t)}X$. In the same way we can compute the directional derivative of a differential form ω since the values $\omega_{|p}, \omega_{|q}$ at different points $p, q \in D$ lie in the dual vectorspace of the tangentspace $\mathbb{R}^2$ of D. Let $X_{|p}, Y_{|p}$ be the values of two vectorfields X, Y then we find with the above local formula also:

$$d\omega_{|p}(X_{|p}, Y_{|p}) = (T_X\omega)_{|p}(Y_{|p}) - (T_Y\omega)_{|p}(X_{|p}).$$

We can now call functions f on a surface *harmonic* iff the differential form $\omega := -df(J\cdot\ )$ is closed and we can formulate the result that exactly the harmonic functions are the real parts of holomorphic (or conformal) ones. But this does not help: While on $\mathbb{C}$ we have no problems finding examples: polynomials, rational functions, power series, ... we cannot write down a single example (different from the constants) on a general surface, one has to appeal to a reasonably hard existence theorem. On the other hand, for *minimal surfaces* the most basic functions turn out to be examples and thereby establish the very close connection between complex analysis and the theory of minimal surfaces in $\mathbb{R}^3$.

The variational definition leads to the geometric formulation:

A surface is a critical point of the area functional iff the trace of its Weingarten map (shape operator) S vanishes. As definition of S we take the Weingarten formula

$$TN = T\Psi \cdot S, \qquad \text{(I reserve, from now on, d for exterior derivatives)}$$

For *sign conventions:* I use the outer normal of the sphere as the natural normal, so that the Gauss map of a sphere is the identity rather than the orientation reversing antipodal map. This is in agreement with the use by analysts who always take the normal field of a level surface ($f = const$) as $N := \operatorname{grad} f/|\operatorname{grad} f|$. Since the principal curvatures of the sphere should be positive I have to take the above sign.

A simple fact from linear algebra:

For a 2-dimensional symmetric endomorphism we have:

$$S^2 = \begin{pmatrix} a & b \\ b & c \end{pmatrix}^2 \overset{claim}{=} \lambda \cdot \mathrm{id}$$

holds either if $a = c,\ b = 0,\ \lambda = a^2$ or if $c = -a,\ \lambda = (a^2 + b^2) = -\det(S)$.

THEOREM. The Gauss map is conformal at p either if the surface is umbilic at p or if the trace of the Weingarten map vanishes.

If we restrict to surfaces of Gauss curvature $K \le 0$ then we can say: The surface is minimal iff its Gauss map is conformal.

PROOF. The scalar product of image vectors under the Gauss map equals $-\det(S)$ times the scalar product given by the first fundamental form $I(\ ,\ )$:

$$\langle TN \cdot v, TN \cdot w\rangle = \langle T\Psi \cdot S \cdot v, T\Psi \cdot S \cdot w\rangle = \langle T\Psi \cdot S^* S \cdot v, T\Psi \cdot w\rangle = -\det(S) \cdot I(v, w)$$

For the last equality we used the linear algebra fact (and ignored positive curvature).

Remark. If we compose the spherical Gauss Map N with orientation reversing stereographic projection then we obtain a holomorphic map to $\mathbb{C}$ - we will call it the holomorphic Gauss map G.

THEOREM. A surface is minimal if and only if its coordinate functions are harmonic on the surface. (As above: f is harmonic iff $\omega = df(J\cdot\ )$ is closed.)

PROOF. We show that the $\mathbb{R}^3$-valued differential form $\omega(v) := -T\Psi(J \cdot v) = -N \times T\Psi(v)$ (see definition of J above) is closed.

$$(T_w\omega)(v) = -T_wN \times T\Psi(v) - N \times \operatorname{hesse}\Psi(w, v).$$

Now $d\omega = 0$ iff the right side is symmetric in v, w. The second term is symmetric. For the first, use the Weingarten formula and check the symmetry of $T\Psi(S \cdot w) \times T\Psi(v)$ on an eigenbasis of S: the symmetry is equivalent to $\operatorname{trace}(S) = 0$ or $\lambda_2 = -\lambda_1$.

The thus established connection with complex analysis will lead, after some preliminaries, to the *Weierstraß representation*.
On any *simply connected* piece of D or on a *simply connected* covering we can integrate the above $\mathbb{R}^3$-valued differential form $\omega(v) := -T\Psi(J \cdot v)$ to a conjugate harmonic map $\Psi^* : D \to \mathbb{R}^3$. Since at every point the image of the differential $T\Psi^*$ is unchanged we have the same Gauss map $N^* = N$. The Weingarten formula gives the new shape operator S^*:

$$TN^* = TN = T\Psi \circ S = -T\Psi \circ J^2 \circ S = T\Psi^* \circ S^*, \text{ hence: } S^* = J \cdot S.$$

The conjugate immersion is therefore another minimal surface. It has the same first fundamental form and the same principal curvatures, but the principal directions are rotated by 45°. These two minimal immersions were constructed to be real and imaginary part of a holomorphic immersion

$$\mathcal{W} := \Psi + \mathbf{i} \cdot \Psi^* : D \to \mathbb{C}^3.$$

We extend the scalar product $\langle \ , \ \rangle$ on $\mathbb{R}^3$ to be *complex bilinear* on $\mathbb{C}^3$ and observe that $\mathcal{W}$ is a so called 'null'-immersion, namely:

$$\begin{aligned}
&\langle T\mathcal{W}(v), T\mathcal{W}(v)\rangle = \langle (T\Psi + \mathbf{i} \cdot T\Psi^*)(v), (T\Psi + \mathbf{i} \cdot T\Psi^*)(v)\rangle \\
&= \langle T\Psi(v), T\Psi(v)\rangle - \langle T\Psi^*(v), T\Psi^*(v)\rangle + 2\mathbf{i} \cdot \langle T\Psi(v), T\Psi^*(v)\rangle \\
&= \langle T\Psi(v), T\Psi(v)\rangle - \langle T\Psi(Jv), T\Psi(Jv)\rangle + 2 \cdot \langle T\Psi(v), T\Psi(Jv)\rangle = 0.
\end{aligned}$$

Vice versa, for every holomorphic null immersion $\mathcal{W} : D \to \mathbb{C}^3$ we have that $\mathrm{Re}\,(\mathcal{W}) : D \to \mathbb{R}^3$ is minimal because we will see shortly that its Gauß map is conformal. We note that this gives in particular the 1-parameter family

$$\Psi_\varphi := \mathrm{Re}\,(e^{-\mathbf{i}\varphi}\mathcal{W}) = \cos\varphi \cdot \mathrm{Re}\,(\mathcal{W}) + \sin\varphi \cdot \mathrm{Im}\,(\mathcal{W})$$

of minimal surfaces, called the *associated family*.

For the next step it is crucial that we have gone from a 2-dimensional real to a 1-dimensional complex description, because only on a 1-dimensional space can one define functions as quotients of differential 1-forms. We take as basis for $\mathbb{C}^3$ an orthonormal basis of $\mathrm{Re}\,(\mathbb{C}^3)$ and write $\mathcal{W} = (\mathcal{W}^1, \mathcal{W}^2, \mathcal{W}^3)$ and we define a function G as follows

$$\begin{aligned}
G :=& \ \frac{-d\mathcal{W}^1 + \mathbf{i} \cdot d\mathcal{W}^2}{d\mathcal{W}^3} \\
=& \left(\frac{+d\mathcal{W}^1 + \mathbf{i} \cdot d\mathcal{W}^2}{d\mathcal{W}^3}\right)^{-1} \qquad \text{since } \langle T\mathcal{W}, T\mathcal{W}\rangle = 0.
\end{aligned}$$

We write $\mathcal{W}$ as the integral of its derivative and, with Weierstraß, express the three component differentials by the function G and $d\mathcal{W}^3$, obtaining the *Weierstraß representation*:

$$\mathcal{W} = \int \left(\frac{1}{2}(1/G - G),\ \frac{\mathbf{i}}{2}(1/G + G),\ 1\right) d\mathcal{W}^3, \qquad \Psi = \mathrm{Re}\,(\mathcal{W}). \qquad \|$$

For every choice of a meromorphic function G and a meromorphic form $d\mathcal{W}^3$, the real part of this (possibly multivalued) *Weierstraß integral* gives a minimal surface; it may not be defined at the zeros and poles of G or the poles of $d\mathcal{W}^3$ and the metric may be degenerate at the zeros of $d\mathcal{W}^3 = d\Psi^3 - \mathbf{i} \cdot d\Psi^3 \circ J$. Abbreviate $dh := d\mathcal{W}^3$ (h for height).

The Riemannian metric (*first fundamental form*) in terms of (G, dh):

$$I(v,v) = \langle T\Psi \cdot v, T\Psi \cdot v\rangle = \frac{1}{2}|T\mathcal{W} \ \cdot v|^2 = \frac{1}{4}\Big(\frac{1}{|G|} + |G|\Big)^2 |dh(v)|^2.$$

Note: If a zero of dh has the same order as a zero or pole of G then these will cancel in the Weierstraß integrand and therefore in the metric. If the order of a zero of dh is larger than the order of a zero or pole of G then the metric degenerates, i.e. Ψ is not an immersion at such a point (called 'branch point'). We will avoid such Weierstraß data.

What is the geometric meaning of the function G?

THEOREM. Stereographic projection of G is the spherical Gauss map N of the minimal surface Ψ (in fact of all the associated Ψ_φ).
In other words: The Weierstraß integral reconstructs a minimal immersion from its meromorphic Gauss map G and its (complexified) height differential.

PROOF. We have to show that $N = (2\text{Re }(G), 2\text{Im }(G), |G|^2 - 1)/(|G|^2 + 1)$ is orthogonal to each tangent vector $X = \text{Re }\Big(\big(\frac{1}{2}(1/G - G),\ \frac{\mathbf{i}}{2}(1/G + G),\ 1\big)dh(e^{\mathbf{i}\varphi}v)\Big)$. Since N is real it suffices to compute the (complex) scalar product:

$$\langle N \cdot (|G|^2 + 1),\ \big(\frac{1}{2}(1/G - G),\ \frac{\mathbf{i}}{2}(1/G + G),\ 1\big)\rangle = \frac{G}{G} - G \cdot \bar{G} + |G|^2 - 1 = 0.$$

Next we compute the *Gauss curvature* K as the volume distortion of the spherical Gauss map. Using conformality this is the square of length distortion. And since we know the length distortion of stereographic projection we compute $|T_vN| = |dG(v)| \cdot (2/(1 + |G|^2))$. Recall $|T_v\Psi| = \frac{1}{2}(\frac{1}{|G|} + |G|)|dh(v)|$. Hence:

$$K = -\frac{|T_vN|^2}{|T_v\Psi|^2} = -\left(\frac{2}{|G| + 1/|G|}\right)^4 \cdot \frac{\left|\frac{dG}{G}\right|^2}{|dh|^2}.$$

Again, because of the conformality, we do not need to compute the matrix of the *second fundamental form* with respect to some basis, we can express directly what it does to vectors. (We always have the polarization identity $b(v + w, v + w) - b(v - w, v - w) = 4b(v, w)$ in mind and therefore compute only the quadratic form $b(v, v)$.) By definition (recall my sign convention)

$$b(v, v) = -\langle \text{Hessian}\Psi(v, v),\ N\rangle = -\langle \text{Re }(\mathcal{W}\,'' \cdot v^2), N\rangle.$$

$\mathcal{W}\,'$ is the Weierstraß integrand. We do not need to differentiate the dh-factor since the other term is orthogonal to N. Hence

$$b(v,v) = -Re\big(\frac{dG}{G}(v) \cdot dh(v) \cdot \big\langle\big(\frac{1}{2}(\frac{-1}{G} - G),\ \frac{\mathbf{i}}{2}(\frac{-1}{G} + G),\ 0\big),\ N\big\rangle\big) = Re\Big(\frac{dG}{G}(v) \cdot dh(v)\Big).$$

The two differential forms $dG/G, dh$ suffice to compute the geometric invariants $K, b(\ ,\ )$.

Application: a tangent vector v is an asymptote direction iff $\frac{dG}{G}(v) \cdot dh(v) \in \mathbf{i} \cdot \mathbb{R}$,
a tangent vector v is a principal curvature direction iff $\frac{dG}{G}(v) \cdot dh(v) \in \mathbb{R}$.

First Examples, defined on $\mathbb{C}$ or $\mathbb{C} \setminus \{0\}$ or $\mathbb{S}^2 \setminus \{1, -1\}$.

Enneper Surface: $z \in \mathbb{C},\ G(z) := z,\ dh := z dz$

Polynomial Enneper: $z \in \mathbb{C},\ G(z) := P(z),\ dh := P(z) dz$

Rational Enneper: $z \in \mathbb{C},\ G(z) := P(z)/Q(z),\ dh := P(z)Q(z) dz$

P and Q are polynomials without common zeros.

Helicoid: $z \in \mathbb{C},\ G(z) := \exp(z),\ dh := \mathbf{i} dz = \mathbf{i}\dfrac{dG}{G}$

Vertical Catenoid: $z \in \mathbb{C} \setminus \{0\},\ G(z) := z,\ dh := dz/z,\ \ (\text{or } G(z) = 1/z)$

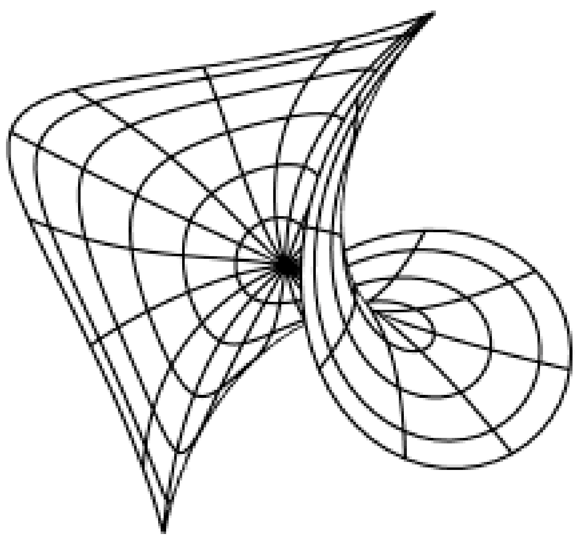
Helicoid: $z \in \mathbb{C} \setminus \{0\},\ G(z) := z,\ dh := \mathbf{i} dz/z$

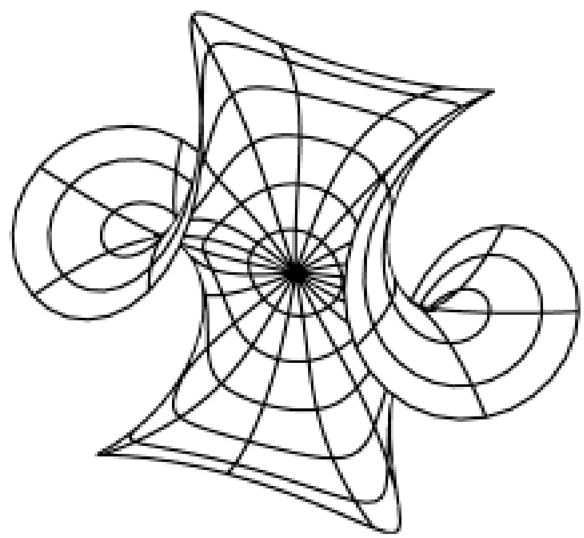
Planar to Enneper: $z \in \mathbb{C} \setminus \{0\},\ G(z) := z^{k+1},\ dh := z^{k-1} dz$

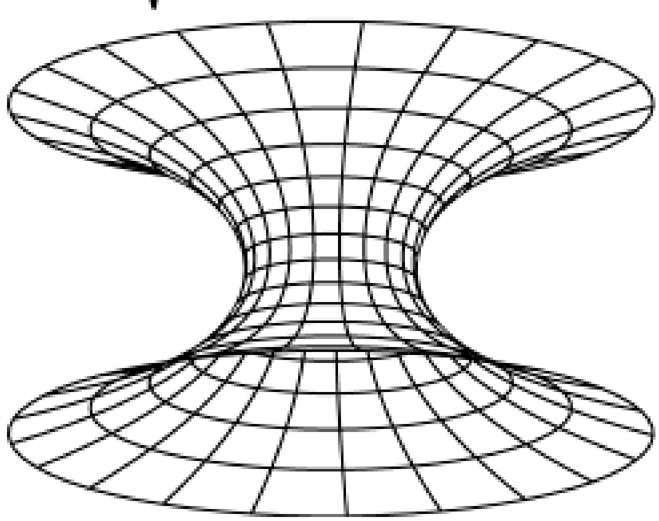
Wavy Catenoid: $z \in \mathbb{C} \setminus \{0\},\ G(z) := (1 + \epsilon \cdot z^k)/z,\ dh := G(z) dz$

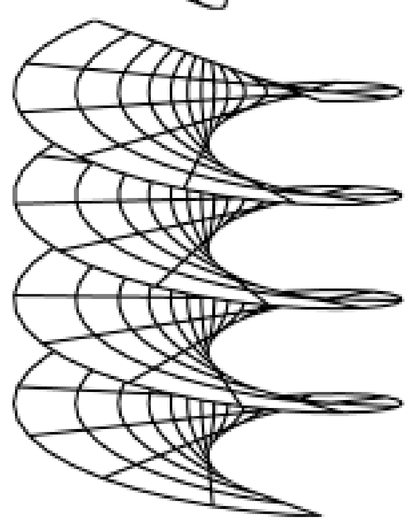
Wavy Plane: $z \in \mathbb{C} \setminus \{0\},\ G(z) := z,\ dh := dz$

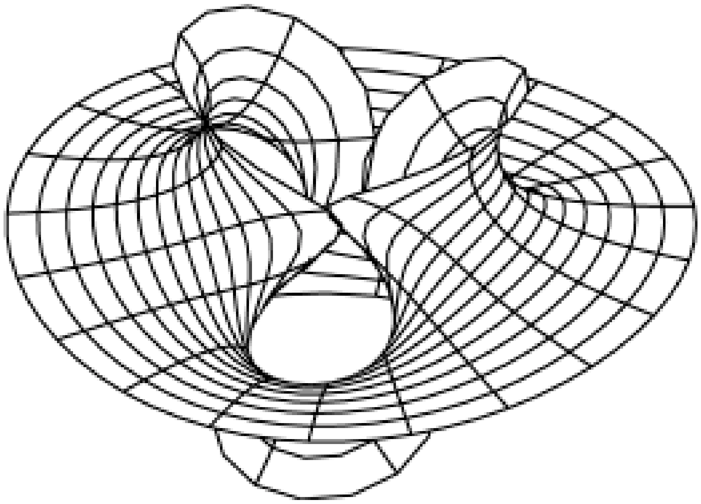
Horizontal Catenoid: $z \in \mathbb{S}^2 \setminus \{1, -1\},\ G(z) := z,\ dh := (z - 1/z)^{-2} dz/z$

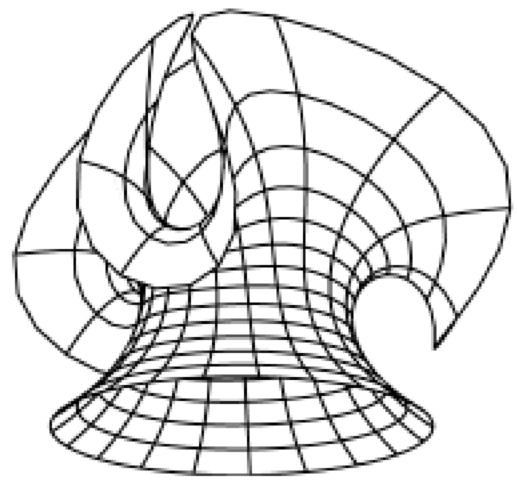


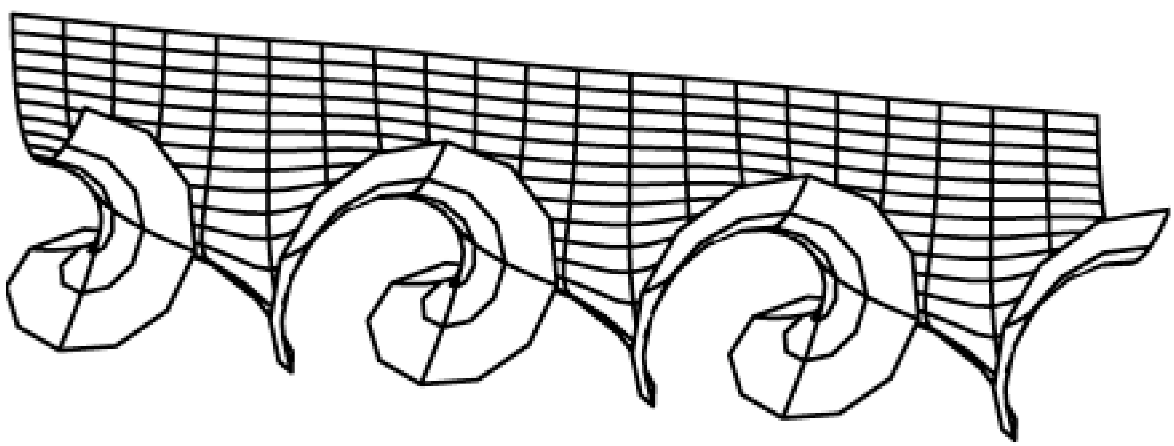












It is convenient to have the above simple criterion for recognizing the asymptote directions and principal curvature directions because they are connected with the symmetries of minimal surfaces as follows.

THEOREM, part I. If a piece of a minimal surface has a straight line segment on its boundary then 180° rotation around this segment is the analytic continuation of the surface across this edge.
Note: Straight lines on a surface are geodesic asymptote lines. If a straight line lies on a minimal surface, then this curve is also a geodesic on the conjugate minimal surface (because both have the same metric). It is no longer an asymptote line but, since $S^* = J \cdot S$, it is a curvature direction. Geodesic curvature lines are planar curves in space and their plane meets the surface orthogonally. This leads to the other half of the symmetry theorem
THEOREM, part II. If a piece of a minimal surface is bounded by a geodesic curvature line then reflection in the plane of the curve is the unique analytic continuation of the surface across the curve.

Before we do the proof let us understand the Frenet theory of curves better and apply it to a minimal surface and its conjugate. Recall that the standard Frenet theory of space

curves does not allow points of curvature zero, in particular does not allow straight lines. On a surface one avoids this by taking as frame: (1) the tangent c' of the curve $\Psi \circ \gamma$, (3) the surface normal $N \circ \gamma$ along the curve, (2) $\eta := N \times c' = T\Psi(J \cdot \gamma')$, called the 'conormal' of the curve. The scalar products of the derivatives with these basis vectors are called respectively:
$\kappa_g := +\langle c'', \eta \rangle = -\langle c', \eta' \rangle$: *geodesic curvature*,
$\kappa_n := -\langle c'', N \rangle = +\langle c', (N \circ \gamma)' \rangle = I(\gamma', S \cdot \gamma')$: *normal curvature*,
$\tau_n := +\langle (N \circ \gamma)',\ \eta \rangle = -\langle N,\ \eta' \rangle = I(S \cdot \gamma', J \cdot \gamma')$: *normal torsion.*

Frenet Equations for Curves on Surfaces.

$$\begin{pmatrix} c' \\ \eta \\ N_\gamma \end{pmatrix}' = \begin{pmatrix} c' \\ \eta \\ N_\gamma \end{pmatrix} \cdot \begin{pmatrix} 0 & \kappa_g & -\kappa_n \\ -\kappa_g & 0 & -\tau_n \\ +\kappa_n & \tau_n & 0 \end{pmatrix}.$$

This says first: for every curve γ in the domain D of the immersion Ψ one can determine the coefficient matrix of this first order system from the geometric invariants $I(\ ,\ ), S$ of the surface. In other words: for every such γ one has the space curve $c = \Psi \circ \gamma$ determined by $I(\ ,\ ), S$. The Gauß Codazzi equations (the integrability conditions) are needed to show that these space curves fit together to a surface.
Secondly, since for geodesics we have $\kappa_g = 0$ and since for conjugate minimal surfaces we have $S^* = J \cdot S$ we conclude a simple relation between normal curvature and normal torsion for corresponding geodesics on such a pair of minimal surfaces:

$$\begin{aligned} \kappa_n &= I(\gamma', S \cdot \gamma') = I(J\gamma', JS \cdot \gamma') = \tau_n^*, \\ \tau_n &= I(J\gamma', S \cdot \gamma') = -I(\gamma', JS \cdot \gamma') = -\kappa_n^*. \end{aligned}$$

In particular, straight line geodesics (asymptote direction) on a minimal surface are planar geodesics (principal curvature direction) on the conjugate surface and vice versa.
Proof of theorem on Symmetries. We may assume, if necessary after rotation, translation and/or conjugation, that the x-y-plane intersects the minimal surface in $\Psi \circ \gamma$, a geodesic curvature line, and that the z-axis lies on the conjugate minimal surface as $\Psi^* \circ \gamma$. We choose the third component function $\mathcal{W}^3$ as a conformal coordinate on the minimal surface. In these coordinates γ is the imaginary axis and the assumption gives $\mathcal{W}(\gamma = \mathbf{i} \cdot \mathbb{R}) \subset \mathbb{R} \times \mathbb{R} \times \mathbf{i} \cdot \mathbb{R} \subset \mathbb{C}$. Thus the standard reflection theorem for holomorphic functions gives:

$$\mathcal{W}(-\bar{z}) = \left(\overline{\mathcal{W}^1},\ \overline{\mathcal{W}^2},\ -\overline{\mathcal{W}^3} \right),$$

which is a reflection in the x-y-plane for the minimal surface Re $(\mathcal{W})$ and a 180° rotation around the z-axis for its conjugate.

After this general introduction my lecture aims at constructing minimal surfaces by using the Weierstraß representation globally. For this we need suitable assumptions and background theorems.
Completeness. We will consider geodesically complete minimal surfaces.
Branch points. We exclude branch points, i.e., we consider *immersed* surfaces.

Huber - Osserman Theorem. A complete immersed minimal surface with finite total curvature $\int |K|\, dA < \infty$ is conformally a *compact* Riemann surface with *finitely many points* removed. The Gauss map G extends as a *meromorphic* function to this compact surface and the complexified height differential dh extends as a meromorphic 1-form.

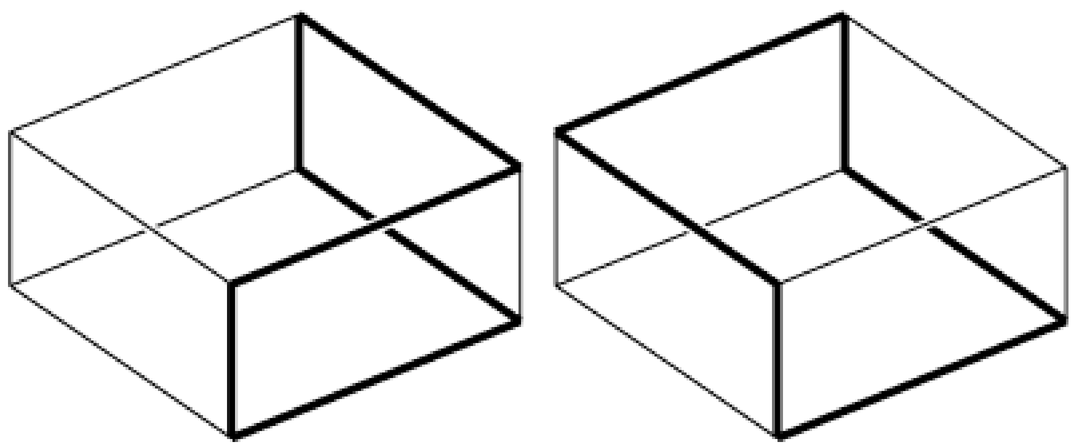
We will not prove this result but it provides strong guiding information.
We recall first the residue theorem, applied to the logarithmic derivative of a nonconstant meromorphic function f; it gives that f assumes all its values equally often. In particular, *f is already determined up to a constant factor by its zeros and poles* (with multipicities), because the quotient of two functions with the same zeros and poles does not assume the values $0, \infty$. Similarly, since the quotient of two meromorphic 1-forms is a function, *the zeros and poles also determine a meromorphic 1-form up to a constant factor*.
Now we apply this to complete minimal surfaces. First note that these can never be compact: the smallest enclosing sphere would touch them at points of positive Gaussian curvature. Complete minimal surfaces extend therefore to infinity, but we can understand how they do this since we can study the Laurent expansion of the Weierstraß data around the points which are missing from the compact Riemann surface. We come back to a discussion of these *ends of minimal surfaces* after we looked at some examples. Applied to the Weierstraß data G, dh of a minimal surface these facts mean: At points of the minimal surface where the normal is vertical we have a zero or pole of the meromorphic Gauss map of some finite order and the form dh must have a zero of the same order there. dh cannot have further zeros and its poles must be at the ends, at the points which are missing from the compact Riemann surface. dh does not need to have a pole at every end since G may have a zero or pole there – see the first fundamental form above. This description is not yet a recipe but it shows the possibility that complete minimal surfaces of finite total curvature might be determined by rather few data.

Educated by the Huber-Osserman theorem we look again at the simple examples above which have Weierstraß data on a sphere with one or two punctures. We will see that it is *much* simpler to make immersed minimal surfaces rather than embedded ones, therefore we are mathematically interested in embedded examples and educationally interested in all of them. Apart from the trivial plane ($G = const$) the catenoid is the only embedded example in the above list. The vertical catenoid has a Gauss map G with a simple pole and a simple zero at the punctures $0, \infty$ and dh has simple poles there. The horizontal catenoid has its zero and pole compensated by zeros of dh and the two double poles of dh make curves which end in ± 1 infinitely long, thus causing $\{-1, +1\}$ not to be in the domain of the Weierstraß integral, while $0, \infty$ are the preimages of those points on this minimal surface which have vertical Gauss map N (or $G = 0, \infty$).
The Enneper surfaces wind around the puncture more than once and they are therefore not embedded. One observes that either Gdh or dh/G has a pole of order larger than 2.
The wavy plane has a so called period: a closed curve on $\mathbb{S}^2$ around the puncture is not mapped to a closed curve. Translation of the intial point to the end point of this not closed curve is a symmetry of the complete minimal surface. Gdh resp. dh/G have simple poles, hence residues at the punctures.
Finally there is one surface with an embedded (and very flat looking) end at 0 and a winding end, similar to the Enneper surfaces, at the other puncture ∞. At the planar

end we observe that dh/G has a double pole (as in the other embedded case, the catenoid ends) at 0 and the Gauss map has a branch point at zero (while G has simple values at the catenoid ends). At the winding end Gdh has a pole of order larger than 2.
It is easy to check from the Laurent expansion of complex functions that this list exhausts all the possibilities. We therefore can see from the Weierstraß data, where the ends are, whether they are embedded or not and also whether the Weierstraß integral does not yield an immersion but has periods at some punctures.

An important result for the construction of minimal surfaces is the solution of the Plateau problem which was obtained by Douglas and independently by Rado in 1932:

Existence of Plateau solutions. To every continuous injective closed curve in $\mathbb{R}^3$ there is at least one minimal surface of disk type which has the given curve as boundary and minimal area (among disk type surfaces with the same boundary).
If the curve has a convex projection then the Plateau solution is unique and is, over the convex interior, a graph and hence embedded. — In the case of the two contours below the convex projection is to one of the vertical faces of the quadratic prism.



The Plateau solutions for these contours can be extended by 180° rotations around the edges until a triply periodic complete surface is obtained. For a better global understanding imagine that $\mathbb{R}^3$ is tesselated by black and by white such prisms in a checkerboard like fashion. If the original piece was in a white prism then one only needs to rotate the prism with the surface to see that all the extensions are in white prisms. Moreover, once one has built a $2 \times 2 \times 2$ block of four black and four white prisms then all further extensions are by parallel translation. In particular, the whole surface is embedded.

By a theorem of R. Krust the conjugate minimal surface of an embedded graph over a convex domain is also embedded (but not with a convex projection). The conjugates of the straight boundary segments are geodesic curvature lines, i.e. planar arcs, and reflection in their planes does the analytic continuation. Since the planes of the symmetry arcs are orthogonal to the corresponding straight edges one gets that these conjugate pieces also lie in orthogonal prisms which are bounded by reflectional symmetry planes of the minimal surface. So again, extension to a complete surface also gives an embedded surface. — While the first extension, by 180° rotations around the edges, rarely gives embedded surfaces, the second extension, by reflections in the planes of geodesic boundary arcs, turned out to be very flexible and many embedded surfaces have been constructed in this way. (I will briefly discuss the Weierstraß representation for such surfaces t the end of these lectures.)

After this excursion to applications of solutions of Plateau problems, combined with conjugation and extension by symmetries, we come back to the finite total curvature surfaces. Why don't I show more embedded ones? If we stay with surfaces parametrized by punctured spheres then they do not exist:

THEOREM OF LOPEZ-ROS. An embedded, minimal, finite total curvature punctured sphere is a plane or a catenoid.

To get practise with the Weierstraß representation we therefore have to be content with a few immersed punctured spheres.

The k-noids of Jorge Meeks, $z \in \mathbb{S}^2 \setminus \{e^{2\pi \mathbf{i} \cdot l/k};\ 0 \le l < k\}$

$$G(z) = z^{k-1}, \quad dh = (z^k + z^{-k} - 2)^{-1} \cdot dz/z.$$

4-noids with two different orthogonal ends, $z \in \mathbb{C} \setminus \{0, -1, +1\}$

$$G(z) = z \cdot \frac{z-r}{1-rz} \cdot \frac{z+r}{1+rz}, \quad dh = \left(1 - \frac{z^2 + z^{-2}}{r^2 + r^{-2}}\right) \cdot (z^2 - z^{-2})^{-2} \cdot dz/z.$$

Two Enneper ends joined by a catenoidal neck, $z \in \mathbb{C} \setminus \{0\}$

$$G(z) = z \cdot \frac{z-r}{1-rz} \cdot \frac{z+r}{1+rz}, \quad dh = \left(1 - \frac{z^2 + z^{-2}}{r^2 + r^{-2}}\right) \cdot dz/z.$$

Three punctures, period closes for tilted ends, $z \in \mathbb{C} \setminus \{-1, +1\}$

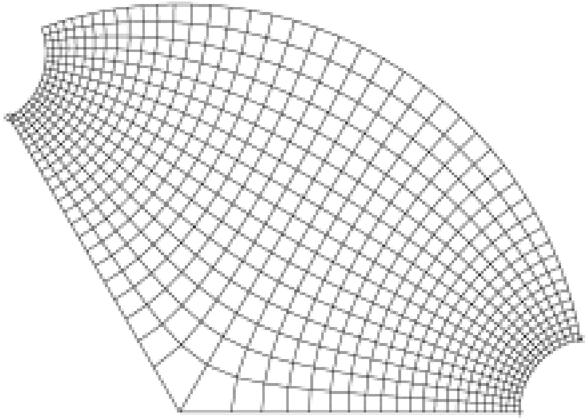
$$G(z) = \rho(z^2 - r^2), \quad dh = \frac{z^2 - r^2}{(z^2-1)^2} dz.$$

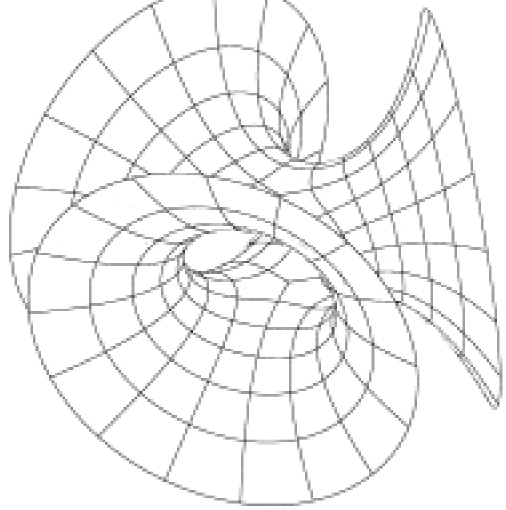
Observe that the zeros and poles of the Gauss map which are not in the list of punctures are compensated by zeros of dh. At the embedded ends, Gdh or dh/G have a double pole and at the Enneper ends they have higher order poles. — In this list we do not have simple poles of Gdh and dh/G. We can allow this and get simply periodic embedded minimal surfaces parametrized by punctured spheres.

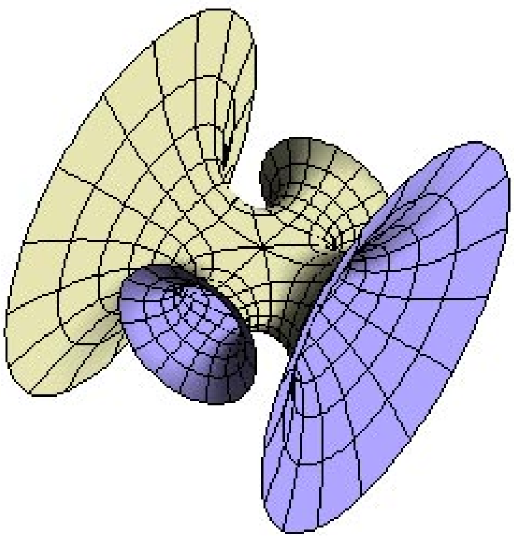
Generalized Scherk Saddle Towers, $z \in \mathbb{S}^2 \setminus \{e^{\pm\varphi} \cdot e^{2\pi \mathbf{i} \cdot l/k};\ 0 \le l < k\}$

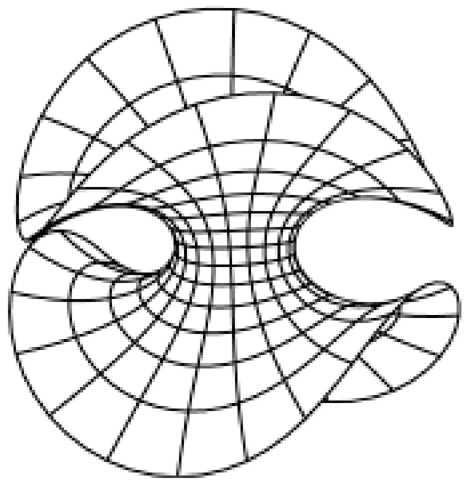
$$G(z) = z^{k-1}, \quad dh = (z^k + z^{-k} - 2\cos k\varphi)^{-1} \cdot dz/z.$$

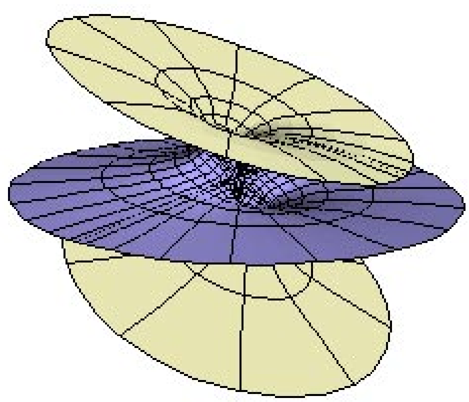
We discuss for the saddle towers how one can read symmetries from the Weierstraß data. The straight lines from 0 to ∞ which are angle bisectors for the direction to the punctures are lines of reflectional symmetry for the metric, hence they are geodesics. Similarly, reflection in the unit circle is an isometry for the metric. In both cases they are also principal curvature lines ($dG(\gamma')/G \cdot dh(\gamma') \in \mathbb{R}$). Therefore we have proved that the horizontal and vertical symmetry lines which the picture seems to have are indeed lines of reflectional symmetry.

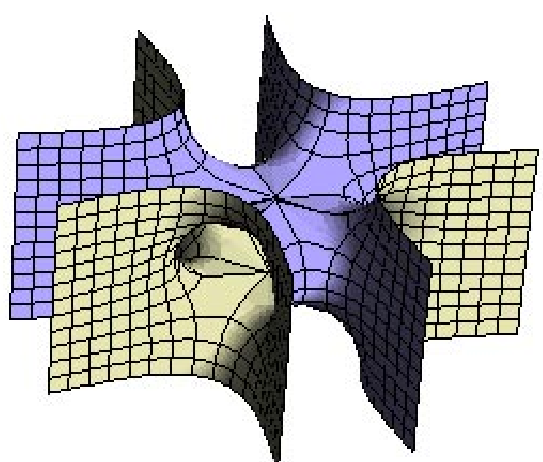












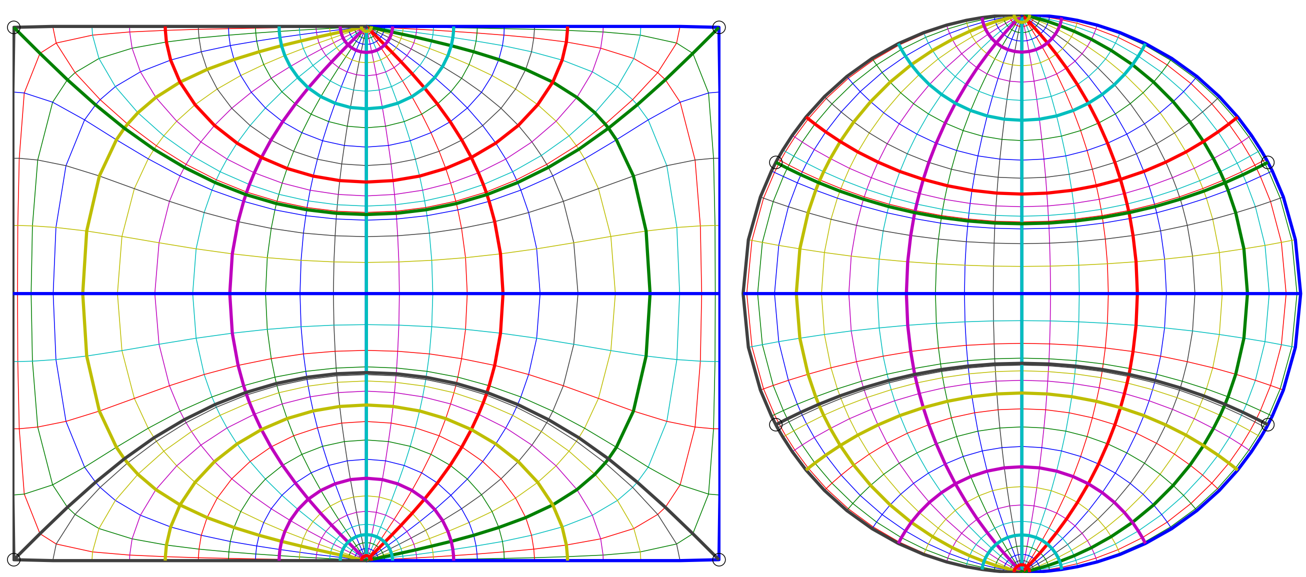
The last example leads to a very rich family of embedded minimal surfaces. If we cut the piece in the figure by its horizontal symmetry plane in half then we obtain a minimal disk which is bounded by horizontal geodesic curvature lines. (The vertical boundaries should be ignored since the surface really extends to infinity.) The conjugate minimal disk is therefore bounded by vertical straight lines; moreover, it projects to a convex polygon which has all its edges of the same length (namely equal to the distance between the horizontal symmetry planes of the original minimal disk). Now vice versa, Jenkins-Serrin have proved that such a graph with boundary values $\pm\infty$ along the edges of a convex polygon exists for example if all the edge lengths are equal. Any such Jenkins-Serrin graph has a conjugate disk which is embedded and bounded by horizontal symmetry lines and these lines lie alternatingly in a top and a bottom symmetry plane. Repeated reflection in these planes extends the conjugate minimal graph to a complete embedded minimal surface with vertical translation period. — If we take the Jenkins-Serrin graph over a square or a rhomic quadrilateral then we can extend it by 180° rotation about the vertical lines to obtain the embedded *doubly periodic Scherk surfaces*. They are conjugate to the ($k = 2$) Scherk saddle towers above.

Where can we look for more embedded minimal surfaces? At least we have seen triply periodic examples which I did not, and indeed cannot, obtain with rational functions. What other meromorphic functions are there?

Consider the Jenkins - Serrin graph over a rectangle with boundary data $0, 0, 0, \infty$. Repeated 180° rotations about the three horizontal edges and the two vertical lines extend this graph to an embedded minimal surface. Rotate this surface so that the long planar looking strips become horizontal, in other words: the limit values of the Gauss map become vertical, what kind of a surface do we see? Do we recognize what function the Gauss map is?

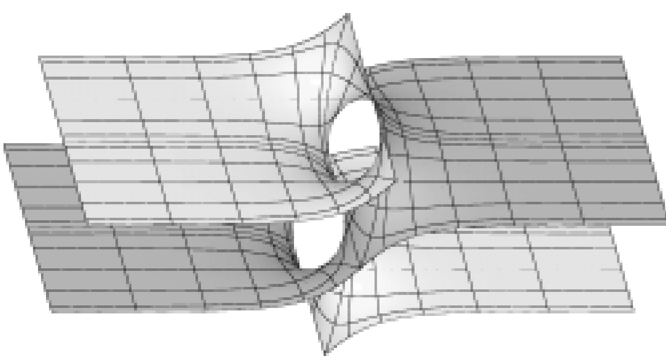
To simplify the picture we take one fundamental domain for the translational symmetries. We can identify the boundary curves, and except for the punctures (where the Gauss map is vertical) we see a surface that is tesselated by rectangles and therefore must be a torus. On this torus the Gauss map has two zeros and two poles. It is therefore a meromorphic 2:1 map from this torus to the Riemann sphere.

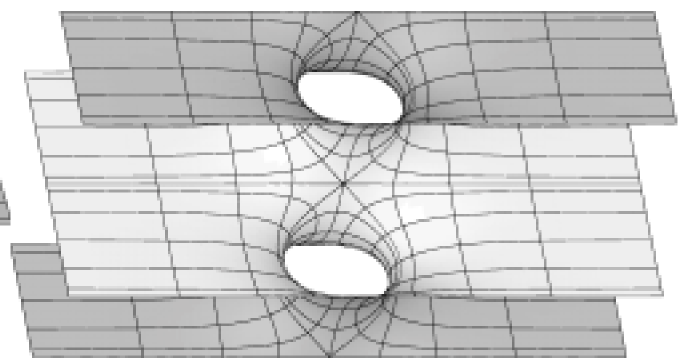
This observation is so close to the standard situation that we can start from scratch: Take a rectangle R and map it via the Riemann mapping theorem to one quarter of the unit disk, in such a way that three vertices of R go to the three corners of the quarter disk. (Such a map is unique.) The fourth point has its image on the circle arc, the precise position depends on the rectangle which we are mapping. Next we use the reflection principle, we extend the map to go from $4 = 2 \times 2$ copies of R to the unit disk, see the picture below:



Use the reflection principle again to get a map from $8 = 2 \times 4$ copies of R to the Riemann sphere and finally from $16 = 4 \times 4$ copies of R to the twice covered Riemann sphere. The values of this map at the left and right boundary and at the top and bottom boundary fit together: we can either consider it as a doubly periodic function on $\mathbb{C}$, or as a conformal $2:1$ map from a torus to the sphere, or as a meromorphic function of degree 2 on the torus.

This function on the torus is our Gauss map G, a so called elliptic function. Finally we need a differential dh which has no zeros nor poles. This does not exist on the sphere (dz has a double pole at ∞, dz/z has simple poles at $0, \infty$). On tori we are lucky: the differential dz of the identity function ($z \mapsto z$) is invariant under translations and therefore well defined on the torus. Maybe we cannot yet compute with this function, but in principle we have the Weierstraß representation of this doubly periodic minimal surface. (View now the right picture as showing the Riemann sphere from the side, with polar coordinates around $0, \infty$.)
If we look back at the initial Jenkins - Serrin graph then its conjugate graph is bounded by geodesic curvature lines. Repeated reflections in their planes extend also this conjugate surface to a complete embedded doubly periodic minimal surface. — The Jenkins - Serrin graph and its conjugate do not reach the boundary of the following pictures; the pictures end in symmetry lines of that graph because they were computed from the Weierstraß representation.





We will only be able to understand minimal surfaces from the global point of view if we learn to see the underlying Riemann surfaces. I will therefore discuss some Riemann surface examples.
Every polyhedron can be made into a Riemann surface by using z^α-coordinates to map a neighborhood of the vertices ('cone points') to disks in $\mathbb{C}$. Clearly all coordinate changes are holomorphic. In particular we have, at first sight, many conformal spheres. This is greatly simplified by the

Uniformization Theorem. Any two conformal spheres are biholomorphic.
Remark. The biholomorphic maps of the Riemann sphere are the Möbius transformations $z \mapsto (az+b)/(cz+d)$. This group is so large that one can choose in the uniformization theorem three arbitrary points on the domain sphere and three arbitrary values on the range sphere to make the biholomorphic map unique.

We know tori for example as $\mathbb{C}$ divided by a lattice group Γ. Any 180° rotation of $\mathbb{C}$

descends to an involution F of the torus $T := \mathbb{C}/\Gamma$. The quotient T/F is again a Riemann surface and Euler's formula for the Euler characteristic shows that the quotient is the sphere. We can therefore specify a meromorphic function on T by specifying the preimages of $0, 1, \infty$. In this way we get many functions on the torus. Each such function has four branch points on the torus, namely the fixed points of the involution F. - The same quotient construction of meromorphic functions, maybe even a bit more intuitive, can be done with tori of revolution in $\mathbb{R}^3$. We get fewer quotient functions since the axis of a $180°$ rotation which intersects the torus in four points has to lie in a symmetry plane.
It is important to note that every meromorphic function f on a Riemann surface provides us with local coordinates near every point, except at the finitely many branch points of f (at simple poles look at $1/f$). To get an atlas one needs at least two functions and they should not have common branch points (at least they should have relatively prime branching orders). The change of coordinates is then given via the implicit function theorem and a relation $F(f, g) = 0$ between the two functions f, g.
I explain for tori how such a relation can be obtained. The Gauss map G which we obtained from looking at a Jenkins - Serrin graph is a map from the torus to the sphere which has two simple zeros and two simple poles. Moreover there are four points where the derivative vanishes and the four branch values are related because of the symmetry of the construction: $B, B^{-1}, -B, -B^{-1}$. Next we define a new function $F := dG/(Gdz)$ where dz is the holomorphic differential without zeros and poles which we met above (it is unique up to a multiplicative constant). Now F has four *simple* zeros at the branch points of G. Therefore we know already that $F, 1/F, G, 1/G$ are a convenient atlas for the torus. But what is the change of coordinates between F and G? The poles of F are at the zeros and poles of G, we have four simple poles. This information says that the functions F^2 and $(G^2 - B^2)(G^2 - B^{-2})/G^2$ have the same zeros and poles (with multiplicity 2) and are therefore proportional. The constant cannot be determined because the holomorphic form dz is only unique up to a constant. Instead we can assume that F^2 and $G^2 + G^{-2} - B^2 - B^{-2}$ are equal and then take $dz = dG/(G \cdot F)$ as the normalized holomorphic form. Then we have:

> The branch values determine the *change of coordinates* via the implicit function theorem applied to the equation between the coordinate functions F and G:
>
> $$F^2 = G^2 + G^{-2} - B^2 - B^{-2}.$$

Now I hope this is enough practise with the concept of a Riemann surface to find the next example of an embedded finite total curvature minimal surface not too difficult.

Before we start looking for a minimal embedded punctured torus of finite total curvature here are results which say the surface will not be very simple:

R. Schoen's Catenoid Characterization.
A finite total curvature embedded minimal surface with exactly two ends is the catenoid.

Hoffman-Meeks Halfspace Theorem. A properly embedded complete minimal surface cannot lie in a halfspace.
A meromorphic embedding must therefore have at least two catenoid ends, one up, one down.

This means we can hope to find a minimal torus with only three punctures. A more symmetric example will be easier to find. Therefore we hope that the surface has two catenoid ends (up and down) and a planar end between them. How could a (non minimal) torus with just one planar end look like? It is easier the other way round: a plane with one handle is the same as a torus with one planar end. Such a surface is simple enough so that it can be imagined with two vertical planes of reflectional symmetry. This excludes all the tori which do not have a reflection symmetric fundamental domain; in fact it also excludes the rhombic tori because each vertical symmetry plane cuts our model torus in two components — as in the case of tori of revolution. At this stage we have three distinguished points on the torus: the intersection of the two symmetry planes meets the surface in three points. We puncture our torus in two of these points, the outside ones. Then we can easily deform these punctures into half-catenoids (up and down). This surface resembles the final minimal surface already strongly.

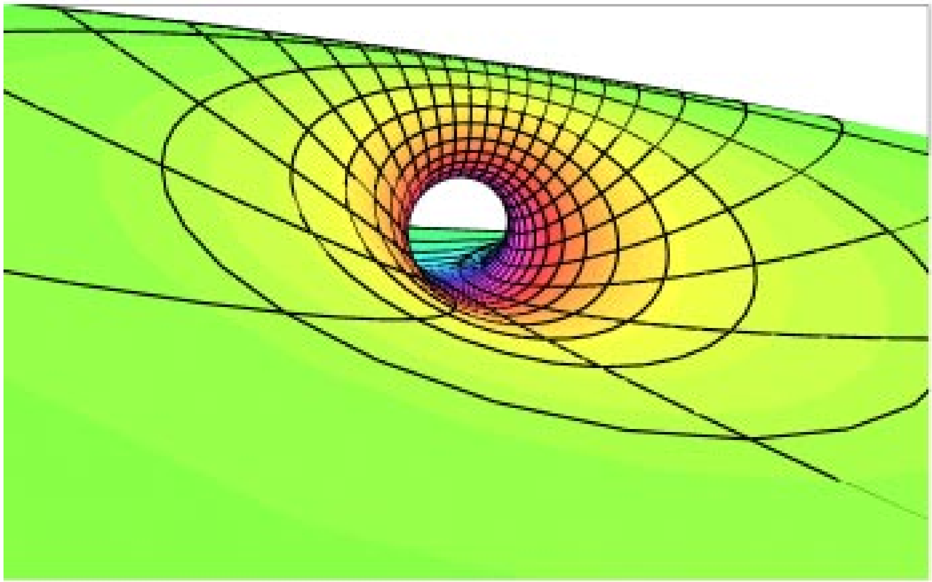
I did not explain the next fact: If one has an embedded minimal surface then the ends are ordered by height and the vertical limit normals point alternatingly up and down. For our model this means: we can assume two simple poles of the Gauss map at the catenoid punctures and at least a double zero at the middle planar end. At the last of the three distinguished points on the symmetry axis the normal is also vertical; our model surface suggests another pole. The simplest Gauss map compatible with the present picture is then of degree three. We need to assume that at the planar end G has a triple zero, because a double zero there and just one other zero not on the symmetry axis is not compatible with the two vertical symmetry planes.

If we draw what we have into the rectangular fundamental domain of our torus we see: three simple poles and one triple zero on the half lattice points. The inverse of the derivative of the Weierstraß $\wp$-function has the same zeros and poles. Therefore we arrived at a candidate for the Gauss map: $G = \rho/\wp'$.

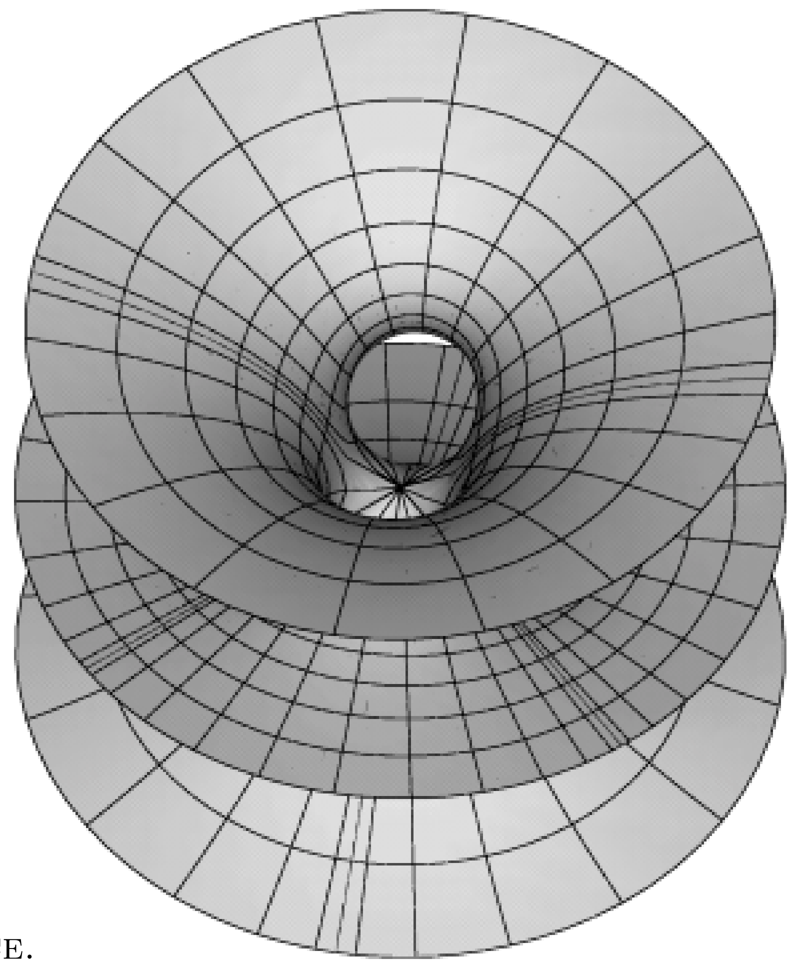
Assuming that this is correct we now need a differential which has simple poles at the catenoid ends and a simple zero at the planar end (namely to give double poles to Gdh or dh/G at embedded ends). And dh needs another zero to compensate the third pole of G (at the last special point on the axis). If we write $dh = H \cdot dz$ then we have met such a function H when we discussed the doubly periodic embedded surfaces. (We also recognize it as $H = (\wp - \wp(branch\ point))/\wp'$.) In other words, up to constant factors we have determined the Weierstraß data of the wanted surface. Scaling dh by a real number only changes the size of the minimal surface; the phase of dh on the symmetry lines (curvature lines) is determined by our earlier discussion. If we multiply the Gauss map by $e^{\mathbf{i}\varphi}$ then we only rotate the surface. The one remaining question is what is the value of the real constant ρ in $G = \rho/\wp'$? This factor is called the "Lopez-Ros parameter" because they made significant use of it in the proof of their theorem quoted above. With the help of the symmetries we find that, in general, the Weierstraß data obtained so far have two horizontal periods. Only on the more symmetric square torus can they be closed by choosing the parameter ρ correctly. This can be done with the intermediate value theorem.

This surface was found in 1982 by C. Costa in his thesis.

Later Hoffman-Meeks found that one can also have other rectangular tori with three punctures minimally embedded; one has to deform the planar middle end into a catenoid end, i.e., one has to split the triple zero of the Gauss map into three simple zeros and the position of the other vertical points is the additional parameter needed to close two periods.



Plane with one handle, or torus with planar end.



The Costa surface.

So far I have only told the story, now we want to learn how the details are done. In particular, we need to understand conformal maps better. First consider how the infinite halfstrip $\{z;\ \mathrm{Re}\, z \le 0,\ 0 \le \mathrm{Im}\, z \le \pi\}$ is mapped by exp to the upper half of the unit disk: The angle 0 at infinity is opened to a 180° angle at 0. We have seen the same phenomenon with the generalizations of Scherk's surfaces: the unit disk with punctures on the boundary is mapped to the Jenkins-Serrin graph over a convex polygon with equal edge lengths and boundary values $+\infty, -\infty$ alternatingly. The Weierstraß integrand has simple poles in the punctures, this explains why the picture looks so similar to the image of a half disk under the map log( ).

We turn to functions on the torus. Consider again the Jenkins-Serrin graph over a rectangle with boundary values $0, 0, 0, \infty$. It is important to see this surface as a conformal rectangle and the Gauss map helps us: the normal rotates on one halfline to infinity until it is perpendicular to the plane of the strip and then it continues to rotate *in the same direction* back from infinity along the other halfline. This indeed allows us to see the surface piece, conformally, as a rectangle and conclude that the complete surface, modulo translations, is a torus, punctured in 4 points. More precisely, the image of the Gauss map of this minimal surface piece is one half of the unit disk, but with a slit on its symmetry line.

The well known Riemann mapping theorem or the lesser known Schwarz Christoffel integrals allow to map a rectangle to a quarter disk so that three 90° corners of the rectangle go to the 90° corners of the quarter disk and the fourth corner is opened to 180°. The reflection principle can be applied to extend this map to a doubly periodic meromorphic map, or, when considered on the quotient torus $\mathbb{C}/\{\textit{period lattice}\}$: as a meromorphic function of degree 2. This is the Gauss map for the doubly periodic minimal surface shown before.

For the Weierstraß data we also need the height differential dh. In this case it should have no zeros or poles since G or $1/G$ have only simple poles, and precisely at the punctures. We are lucky: such a differential is the only one which we know on a torus, namely the translation invariant dz. So now we understand the Weierstraß data in principle, but we cannot yet compute since we cannot evaluate G as a doubly periodic function. For this purpose we derive a differential equation. – Above we derived an equation between two functions F, G on the torus and interpreted it as describing the change of coordinates between the 'coordinate functions' F, G. That derivation, with the normalization at the end, says:

$$G' = \frac{dG}{dz} = FG, \qquad (FG)^2 = G^4 + 1 - G^2 \cdot (B^2 + B^{-2}).$$

This differential equation we can solve in $\mathbb{C}$ provided we know how to deal with the complex square root of a polynomial. I tried to teach you to program an analytic continuation of the square root. Such a continuation is not possible on a grid which contains zero or infinity. But of course, F assumes these values. How can this be handled?

Consider as a simpler case the first order differential equation for sin: $\sin' = \sqrt{1 - \sin^2}$. It has difficulties near the points with $\sin^2 = 1$. One further differentiation gives a harmless second order equation: $\sin'' = -\sin$. This trick also works for our equation for G. Let P be any polynomial then:

$$(G')^2 = P(G) \qquad \text{implies} \qquad G'' = \frac{1}{2} P'(G).$$

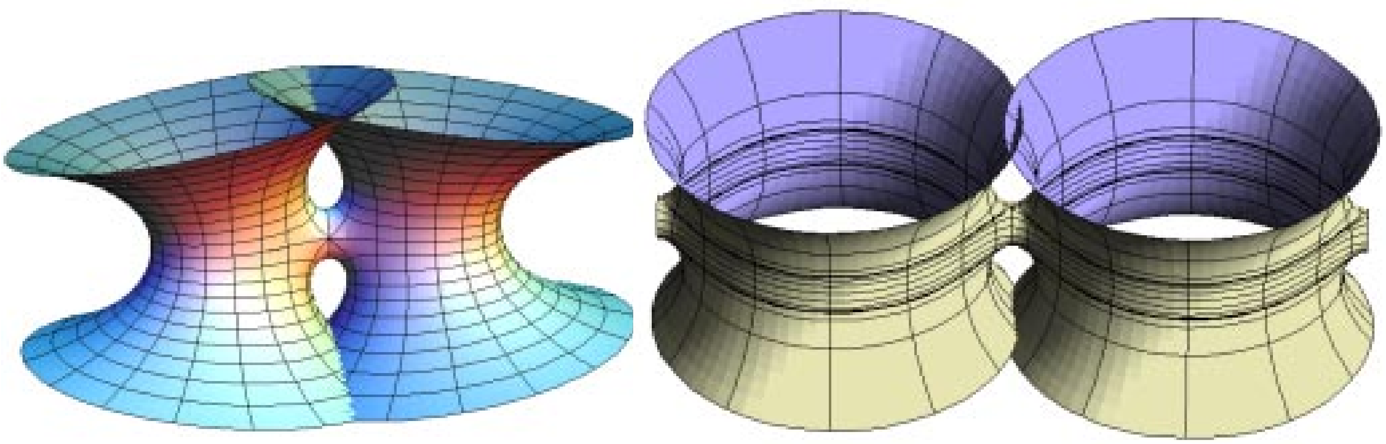
This information is enough to actually compute the Weierstraß integral.

But it is instructive to see how this can be further improved. One reason is esthetic: the picture looks better if the parameter lines are obtained from polar coordinates around the puncture (consider how log maps a half disk to a strip). Of course one needs to generate such grids as preimages of polar coordinate grids in the range of the Gauss map. The other reason is mathematically interesting because it is not really necessary to evaluate elliptic functions to do the Weierstraß integration. Instead one has to understand that the integration of a differential form can be done in any coordinate system. So, instead of integrating $(1/G - G)(z)dz$ we substitute $dz = dG/(G \cdot F(G))$. Now we simply integrate the square root of a polynomial over a polar coordinate grid in the range of G:

$$\boldsymbol{\mathcal{W}} = \int \left(\frac{1}{2}(\frac{1}{G} - G),\ \frac{i}{2}(\frac{1}{G} + G),\ 1 \right) \cdot \frac{dG}{\sqrt{P(G)}}.$$

In the 19^{th} century H. A. Schwarz used the Weierstraß representation in similar ways.

With this experience we can now construct surfaces parametrized by tori. Except that we need to guess some shapes. Let us look at the skew4noid family near the boundary of its parameter range. There, surfaces look like two catenoids which are joined by a handle (which distorts them slightly).



Assume another surface family started as a catenoid but grew two handles, in symmetric positions, one on each side. Such a surface clearly is a torus with two punctures, a rectangular torus because of the symmetries. Its Gauss map has two zeros and two poles, on intersections of vertical symmetry lines. Such functions we have already met. The differential needs simple zeros at one zero-pole pair and simple poles at the other. If we write $dh = Hdz$ then H is also one of the degree 2 functions we know. So, except for getting the constants right, we know we are done. Note that there will not be a period problem at the punctures because two different vertical symmetry planes go through the punctures: a period would have to be orthogonal to both of them (compute the period on a symmetric curve around the puncture). And the small handle also has no period because the symmetry planes cut the curve around the handle into four congruent parts.
For the discussion of the constants, map $1/16^{\text{th}}$ of the rectangular fundamental domain to the quarter circle $0, 1, \mathbf{i}$ so that in both cases the same two vertices of the rectangle go to $0, \mathbf{i}$, but in the case of the Gauss map G the branch value B_G is on the unit circle while

the branch value $B_H \in [0,1]$ for the function H. This implies that the functions $G+1/G$ and $H+1/H$ have the same zeros and poles and are therefore proportional. The factor is obtained at that vertex that is mapped to 1 (which gives us the coordinate change between G and H):

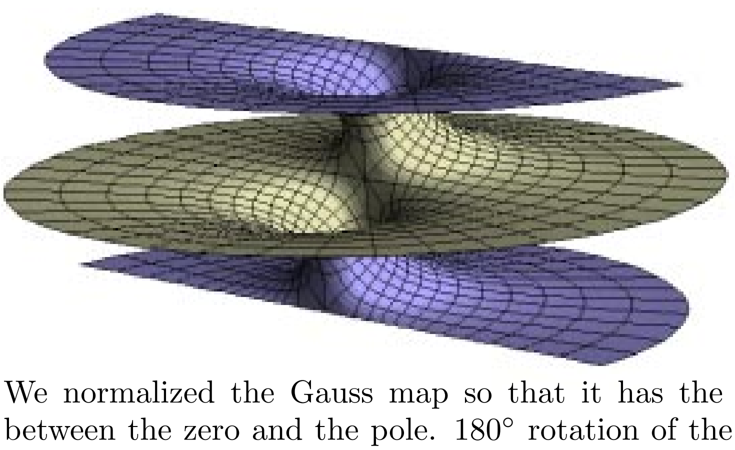
$$\frac{G+1/G}{2} = \frac{H+1/H}{B_H+1/B_H} \quad \text{or} \quad \frac{G+1/G}{B_G+1/B_G} = \frac{H+1/H}{2}.$$

Again by comparing zeros and poles we find $H'/H \sim G-1/G$; note that here the constant cannot be determined because the size of the domain torus is arbitrary. We can put $H'/H = G-1/G,\ dz = dH/H',\ G = G(H)$. This is explicit enough to compute this *fence of catenoids.*

For another example look again at the conjugate surface of the Jenkins-Serrin graph at the beginning of the torus discussion. The period translations are generated by 180° rotations around pairs of lines on the surface; there are translations which translate the (horizontal) plane-like portions into themselves and other translations which map one level to another. Now imagine that we could increase the distance between the horizontal lines to infinity. What would a limit surface look like? It would look like a stack of planes and each plane has one handle to the next plane up (and therefore one handle — observed as tunnel — to the next plane down). In other words, a cylinder with planar ends.

Divide out the translations to get a torus with two punctures and no other vertical normals. Around each handle we imagine a curve with horizontal normals. So the Gauss map is of degree 2 and $dh = dz$. If we call the Gauss map of the previous example G_C then the new Gauss map is $G := -\mathbf{i}(G_C - B_G)/(G_C + B_G)$ and $dh = dz = dG'_C/G'_C = dG/G'$. (Because of its only (double) pole, G is close to the Weierstraß $\wp$-function, $G = a \cdot \wp + b$.)

These Weierstraß data were chosen to give a surface with the desired planar ends and with the expected symmetry lines (coming from the symmetries of the $\wp$-function). How about periods? The punctures have no periods since $Gdh, dh/G, dh$ have all residues $= 0$. One generator of the torus is mapped onto the vertical symmetry line which joins one pucture to the next; through each handle run two such curves. But why is the other generator mapped to a closed curve, or in other words, why are the two vertical symmetry lines through the same handle not only in parallel planes, why are they *in the same plane*?



This is

Riemann's Minimal Surface.

It exists for every rectangular torus. Its conjugate is another example. The level lines parallel to the ends are circles. From this fact Riemann deduced its representation. Its Gauß map has just one double zero and one double pole.

We normalized the Gauss map so that it has the values $\pm\mathbf{i}$ at the diagonal midpoints between the zero and the pole. 180° rotation of the torus therefore changes the values as $G \mapsto -1/G$. This implies, that the first and second fundamental forms of our Weierstraß data are *not* changed, the rotation around each corresponding normal is a rigid rotation of the minimal surface. In particular, there is *only one* vertical symmetry plane.

With the added experience of the last two examples the earlier description of the Costa surface is complete except for the discussion of the periods. Observe the following general fact: If a homotopy class has a representative which is symmetric relative to a symmetry line of the minimal surface then we compute the period of the homotopy class on such a representative. This gives

- if the symmetry line is a *planar geodesic* then the period is perpendicular to this plane;
- if the symmetry line is a *straight line* then the period is perpendicular to this line;
- if the representative is symmetric with respect to a rotation about a normal then the period is perpendicular to this normal.

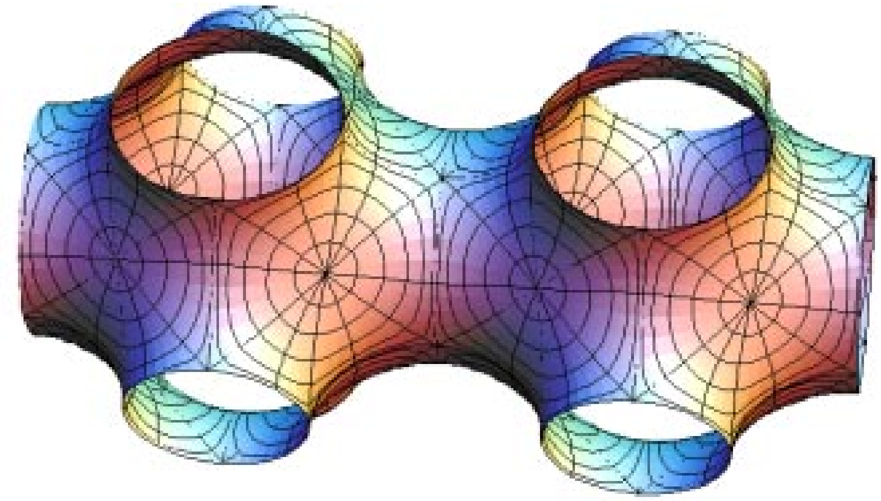
This shows that the Costa surface has at most horizontal periods. There is a convenient reformulation of the period closing condition. For any closed curve γ on the Riemann surface we have

$$\mathrm{Re}\left(\int_\gamma \Big(\frac{1}{2}(1/G - G),\ \frac{i}{2}(1/G + G)\Big) dh\right) = 0 \quad \text{if and only if} \quad \int_\gamma \frac{1}{G} dh = \overline{\int_\gamma G dh}.$$

If, as in the case of the Costa surface, the Gauss map has been determined up to the Lopez-Ros parameter ρ, then choose first $\rho = 1$ and check whether the two integrals are real and positive. If this is true then $\rho^2 = \int_\gamma \frac{1}{G_1} dh \,/\, \overline{\int_\gamma G_1 dh}$ is the correct choice of ρ, $G = \rho \cdot G_1$. For the Costa surface this can be checked without computation.

We used the discussion of minimal surfaces parametrized by tori to introduce the concept of Riemann surfaces. In the context of the Weierstraß representation this is natural since the Weierstraß integrand is built from geometrically defined functions and forms. Still, for tori one could have avoided the new concept since functions and forms on tori can be adequately dealt with via doubly periodic functions in $\mathbb{C}$. For higher genus cases Riemann surfaces become more essential. It is a theorem for arbitrary compact Riemann surfaces that there always exist two meromorphic functions f, g which, when considered as coordinate functions, give an atlas for the surface; usually $\{f, 1/f, g, 1/g\}$ is such an atlas, sometimes products $f^m g^n,\ m, n \in \mathbb{Z}$ have to be added. The change of coordinates is specified via the implicit function theorem by an algebraic relation $P(f, g) = 0$ between these functions. To use this approach in a constructive way one has to find two suitable functions on the surface, together with enough information about these functions so that one can determine the algebraic relation. (Recall that two functions on a *compact Riemann surface* agree if they have the same zeros and poles and agree at one more point.) In applications to minimal surfaces one usually knows enough about the Gauss map G. Quite generally $H := dG/(Gdh)$ could serve as a second coordinate function, but its degree is so high that often one cannot find an algebraic relation $P(G, H) = 0$. An approach which usually leads to relations of reasonably low degree depends on symmetries of the surface. I describe two variations. First, if symmetry lines divide the surface into simply connected domains then it may be possible to use the Riemann mapping theorem to map a tile of the surface to a tile of the Riemann sphere in such a way that extension by reflection defines a meromorphic function. Secondly, the Riemann surface may have a symmetry group such that the quotient surface by this group is a conformal sphere (and by the uniformization theorem is the Riemann sphere); the quotient map, therefore, is already a map to the sphere, but to specify a function one has to eliminate the freedom of the Möbius group; this is done by specifying three values, for example, one specifies the points which are mapped to $0, 1, \infty$.

We illustrate this with a triply periodic minimal surface found by of H. A. Schwarz.
The surface was named Schwarz ‘P’-surface by A. Schoen, who discovered more triply periodic surfaces.



This surface is triply periodic. A translational fundamental domain is bounded by convex symmetry lines which lie on the faces of an orthogonal prism over a rectangle. If we ignore scaling then we have a 2-parameter family of such surfaces and we will derive a 2-parameter family of Weierstraß data. We identify the boundaries of a fundamental piece by translations to get a Riemann surface of genus 3. It is tesselated by eight 90° hexagons (check the Eulercharacteristic: $\chi = F - E - V = 8 - 24 + 12 = -4$).
We know a lot about the Gauss map: Normals at opposite vertices of a hexagon are parallel and the three different normals (at the vertices of one hexagon) are orthonormal. Each hexagon is therefore mapped 2:1 onto a spherical triangle with 90° angles. The eight image triangles tesselate the sphere. The Gauss map has degree 2. The Gauss map has a branch point in each hexagon and the eight branch values lie in different octants of the sphere.
To get a second function imagine the hexagons colored black and white in checkerboard fashion. Map (via the Riemann mapping theorem) a white hexagon to the unit disk, for normalization send the midpoint (= branch point of G) to 0 and send the (opposite) points with vertical normal to ± 1. Clearly, the map extends by reflection to a degree 4 meromorphic function S.
To derive a relation observe that the branch values of G are related, they can be arranged as $\{B_j, \bar{B_j}\} = \{B, -B, 1/B, -1/B, \bar{B}, -\bar{B}, 1/\bar{B}, -1/\bar{B}\}$. The equated two functions have the same zeros and poles and agree at the vertical points of G, dh is explained below:

$$S^2 = \frac{\prod(G - B_j)}{\prod(G - \bar{B}_j} = \frac{G^2 + 1/G^2 - (B^2 + 1/B^2)}{G^2 + 1/G^2 - (\bar{B}^2 + 1/\bar{B}^2)}, \qquad dh = (S - 1/S) \cdot \frac{dG}{G}.$$

This equation defines a 2-parameter family of Riemann surfaces. The differential dh should not have poles (since the minimal surface modulo translations is compact) and it should have simple zeros at the simple zeros and poles of G. On this surface we know nothing like the dz on the torus, therefore we have to write dh as a multiple of dG/G or of dS/S (we use the logarithmic differentials to keep the symmetry between zeros and poles). dG/G has simple zeros at the centers of the hexagons and has simple poles at the vertical points of G; since ± 1 are branch values of S the function $S - 1/S$ has simple poles at the hexagon centers and double zeros at the poles of dG/G. This explains dh.