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SOME APPLICATIONS OF SHORT CORE MODELS*

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We survey the definition and fundamental properties of the family of short core models, which extend the core model K of Dodd and Jensen to include α-sequences of measurable cardinals ($\alpha \in \mathrm{On}$). The theory is applied to various combinatorial principles to get lower bounds for their consistency strengths in terms of the existence of sequences of measurable cardinals. We consider instances of Chang's conjecture, 'accessible' Jónsson cardinals, the free subset property for small cardinals, a canonization property of ω_ω, and a non-closure property of elementary embeddings of the universe. In some cases, equiconsistencies are obtained.

0. Introduction

A major theme in axiomatic set theory is the ranking of consistency strengths of combinatorial principles by the linear scale of large cardinal axioms. Typically, a forcing construction is employed to extend a model of a large cardinal property to a model of the combinatorial property considered, whereas, given the principle, one seeks for large cardinals within inner models. *Constructible* models of set theory—Gödel's model L of constructible sets, Silver's L^μ for a measurable cardinal [18], Mitchell's L^F for a coherent sequence of measures [11, 12], the core model K of Dodd and Jensen [5] and its generalizations—provide natural inner models for large cardinals up to high orders of measurability. If an ordinal is, say, measurable in some inner model, then it is measurable within some Silver model L^μ. Thus in applications of inner models it is often advantageous to restrict to well-structured 'L-like' models right away.

In my doctoral dissertation [9] I studied the family of *short core models* which, roughly speaking, approximate inner models with α measurable cardinals for some ordinal α. The covering and condensation properties of short core models were applied to obtain information on the consistency strengths of certain instances of Chang's conjecture, the Jónsson property and the free subset property. The family of short core models which forms just a small subfamily of the general core models for sequences of measures as studied by Mitchell [13, 14] was chosen for several reasons: Since in these models every ordinal carries at most one measure the generalized fine structure theory of Dodd [4] can be

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applied (if there were measurable cardinals of Mitchell order $\geqslant\omega$, a much more complicated fine structure is necessary [6]). Short core models have long stretches of ordinals without measures where the models behave very L-like. Short core models satisfy covering theorems with Prikry sequences which are known to fail for large core models. Apart from results and techniques which undoubtedly will be generalized to higher core models we also obtained some equiconsistencies of the strength "there are α measurable cardinals".

The aim of the present article is to give an overview of short core model theory and, taking fundamental properties of mice and short core models for granted, to prove our applications in detail. The theory of my thesis has been improved and I also include results on canonical form properties (Shelah [17]) and on a non-closure property of elementary embeddings as considered by Sureson [20]. Section 1 informally shows how core models canonically arise if one wants to prove the existence of inner models with (many) measurable cardinals. We sketch how such models might be obtained from strong combinatorial principles. Section 2 gives an outline of the coarse (= non-fine-structural) theory of mice. It is possible to a certain extent—and will be done in this paper—to define and use short core models without any explicit fine structure, the fine-structural details being nicely encapsulated within the fundamental theorems on short core models. These theorems are presented in Section 3. The remaining sections contain the applications of our theory which we indicate here by typical instances (the combinatorial principles will be defined later):

Theorem 1. *The Chang property $(\omega_3, \omega_2) \Rightarrow_\omega (\omega_2, \omega_1)$ implies that for every $\alpha \in \mathrm{On}$ there is an inner model with α measurable cardinals* (4.3, 2.14).

Theorem 2. *If λ is a Jónsson cardinal such that $\lambda = \omega_\xi$, $\xi < \lambda$, or such that λ is regular but not weakly hyper-Mahlo, then for every $\alpha \in \mathrm{On}$ there is an inner model with α measurable cardinals* (5.2, 5.4, 2.14).

Theorem 3. *The theories "ZFC + there is a Jónsson cardinal of cofinality ω_1" and "ZFC + there are ω_1 measurable cardinals" are equiconsistent* (5.7).

Theorem 4. *The free subset property $\mathrm{Fr}_\omega(\omega_{\omega_1}, \omega_1)$ is equiconsistent to "There are ω_1 measurable cardinals"* (6.7).

Theorem 5. *Assume that $\langle \omega_{k(n)} \mid n < \omega \rangle$ has a $\langle 3 \mid n < \omega \rangle$-canonical form for $\{\langle \overbrace{2, 2, \ldots, 2}^{i\text{-times}} \rangle_2^{2i} : i < \omega\}$, where the sequence $\langle \omega_{k(n)} \mid n < \omega \rangle$ has supremum ω_ω. Then for every $\alpha \in \mathrm{On}$ there is an inner model with α measurable cardinals* (7.2, 2.14).

Theorem 6. *If there is an elementary embedding $j : V \to M$, with critical point κ*

such that ${}^{\omega}M \subset M$ *and* ${}^{\kappa}M \not\subset M$, *then for every* $\alpha \in \mathrm{On}$ *there is an inner model with* α *measurable cardinals* (8.1, 2.14).

The combinatorial principles considered could be weakened while still yielding the same consequences. In Theorem 1, e.g., a weak Chang property as defined in [7] would give the same conclusion.

This article requires some acquaintance with constructibility in terms of the relativized J_α-hierarchy and of iterated ultrapowers. Standard set-theoretical notation will be used throughout.

1. Motivation

Assume we are to define an inner model with α measurable cardinals, that is, α should be the ordertype of the set of measurable cardinals. The Mitchell models L^F are constructible models of this kind, and we have to find a sequence F of filters such that:

(∗) $L^F \models$ "F is a sequence of measures on measurable cardinals and the ordertype of measurables is α".

The subsequent informal argument will provide us with a 'local' criterion for (∗) to hold. It will allow the construction of F by recursion on the ordinals in the domain of F (we stipulate that $\mathrm{dom}(F) \subset \mathrm{On}$ and $F(\kappa)$ is a filter on κ for every $\kappa \in \mathrm{dom}(F)$).

Let us analyse the situation where (∗) fails. To facilitate our reasoning we assume that F is *countably complete* (i.e., for every $\kappa \in \mathrm{dom}(F)$ and $\{X_i \mid i < \omega\} \subset F(\kappa): \bigcap \{X_i \mid i < \omega\} \neq \emptyset$), and that $\alpha < \min \mathrm{dom}(F)$. Since (∗) is a Π_1-condition there will be a $\beta \in \mathrm{On}$ such that:

$$J_\beta[F] \models \text{“}F \text{ is a sequence of measures”}, \quad \text{but}$$
$$J_{\beta+1}[F] \models \text{“}F \text{ is } \textit{not} \text{ a sequence of measures”}.$$

In $J_{\beta+1}[F]$ there is a $\kappa \in \mathrm{dom}(F)$ and a $c \subset \kappa$ which codes a counterexample to $F(\kappa)$ being a normal measure on κ (c could be a non-measured subset of κ, or it could code a regressive function which is not constant almost everywhere). In some weak sense, c has to be definable over $M := J_\beta[F]$ (using F), but to avoid fine-structural arguments we assume here that c is Σ_1 over M. We shall locate c within a naturally defined constructible model.

Let $U := F \restriction \kappa$. A *U-mouse* is a structure $N = J_\gamma[G, U]$ such that:

(i) $N \models$ "G is a sequence of measures on ordinals $> \sup(U)$".

(ii) $\mathrm{opt}\,\mathrm{dom}(U \cup G) \leq \min \mathrm{dom}(U)$ if $\mathrm{dom}(U) \neq \emptyset$ and $\mathrm{otp}(\mathrm{dom}(G) \cap \eta) < \min \mathrm{dom}(G)$ for all $\eta < \omega\gamma$ if $\mathrm{dom}(U) = \emptyset$.

(iii) N is iterable by the measures in G in terms of iterated ultrapowers.

The *low part* of the U-mouse N is defined to be $\mathrm{lp}(N) := H^N_\lambda$, if $\lambda = \min(\mathrm{dom}(G))$ exists, and $\mathrm{lp}(N) := N$ otherwise.

$K[U] := \bigcup \{\mathrm{lp}(N) \mid N$ is a U-mouse$\}$ is an inner model of ZFC. In case $K[U] \models$ "U is a sequence of measures", $K[U]$ is called a *core model*.

Now, by the countable completeness of F, $M = J_\beta[F \setminus U, U]$ is a U-mouse. Iterate M by all the measures in $F \setminus U$ μ-many times for μ a sufficiently big regular cardinal; let $N = J_\gamma[G, U]$ be this iterate. Indiscernibility arguments show that N can be extended to a U-mouse $N^+ = J_{\gamma+1}[G^+, U]$, $G^+ \supset G$. Since the iteration maps are Σ_1-elementary, c is $\Sigma_1(N) \subset N^+$. So $c \in \mathrm{lp}(N^+) \subset K[U]$. (If the definition of c over M is more involved than Σ_1, M has to be iterated in a more elaborate, fine-structure preserving way.) So if F is not a sequence of measures in L^F, then $F(\kappa)$ does not measure $\mathcal{P}(\kappa) \cap K[F \restriction \kappa]$ for some κ. This implies the following criterion for a countably complete filter sequence to satisfy $(*)$:

(†) Assume that for all $\kappa \in \mathrm{dom}(F)$, $F(\kappa)$ is a normal measure on $\mathcal{P}(\kappa) \cap K[F \restriction \kappa]$. Then $L^F \models$ "F is a sequence of measures".

This can be extended to:

(††) Assume that for all $\kappa \in \mathrm{dom}(F)$, $F(\kappa)$ is a normal measure on $\mathcal{P}(\kappa) \cap K[F \restriction \kappa]$. Then $K[F] \models$ "F is a sequence of measures". So $K[F]$ is a core model, and all the models $K[F \restriction \kappa]$ are core models, too.

(††) is useful for constructing a good sequence F by recursion on $\mathrm{dom}(F)$.

We shall now briefly describe a strategy to show that a strong combinatorial principle P implies the existence of an inner model with α measurable cardinals. If it is possible to apply (††) α times we are done of course. Otherwise, application of (††) leads to a *maximal* core model $K[U]$ where the measure sequence U cannot be properly end-extended. Every $x \in K[U]$ is in the low part of some U-mouse M, the low part consisting exactly of those elements of M which are not altered by iterations. M may be iterated up to a U-mouse $M^* = J_\beta[C, U]$ where C is a sequence of closed unbounded filters on regular cardinals. So any set $x \in K[U]$ and indeed any subset of $K[U]$ may be considered within some M^* with a rather simple filter sequence. It then becomes conceivable that $K[U]$ is an L-like inner model.

The maximality of $K[U]$ implies that $K[U]$ 'covers' the universe to a degree, which imposes traces of constructibility upon the set-theoretical universe. If our principle P now is sufficiently 'non-constructible' this leads to a contradiction, and the definition of filters according to (††) must be possible α times as required.

This vague plan will be realized in various ways in the second half of this paper.

2. Iterable premice

We work with the relativized J-hierarchy $J_\alpha[\vec{A}]$, $\alpha \in \mathrm{On}$, where the $\vec{A}$ are used as predicates in the recursive definition of the hierarchy. $J_\alpha[\vec{A}]$ will denote the set

$|J_\alpha[\vec{A}]|$ as well as the $\in$-structure $\langle |J_\alpha[\vec{A}]|, \in \upharpoonright |J_\alpha[\vec{A}]|, \vec{A} \upharpoonright |J_\alpha[\vec{A}]| \rangle$. If $M = J_\alpha[\vec{A}]$, then $<_M$ denotes the *canonical well-ordering* of M. For details the reader is referred to Devlin [2].

A class D is *simple* if (i) every $x \in D$ is of the form $x = \langle \kappa, a \rangle$ where $\kappa \in \mathrm{On}$ and $a \subset \kappa$, and (ii) if $\langle \kappa, a \rangle \in D$ then $\langle \kappa, \kappa \rangle \in D$. A filter sequence as in Section 1 can be viewed as a simple predicate. For D simple define: $\mathrm{dom}(D) := \{\kappa \mid \langle \kappa, \kappa \rangle \in D\}$, $D(\kappa) := \{a \mid \langle \kappa, a \rangle \in D\}$ for $\kappa \in \mathrm{dom}(D)$, and $D \upharpoonright X := \{\langle \kappa, a \rangle \in D \mid \kappa \in X\}$.

U is a *measure* on κ if U is a non-principal, κ-complete, normal ultrafilter on κ. F is a *sequence of measures* if F is simple and $F(\kappa)$ is a measure on κ for every $\kappa \in \mathrm{dom}(F)$.

2.1. Definition. Let D be simple. $M = J_\alpha[F, D]$ is a *premouse over D* provided:

(i) F is simple.

(ii) $\sup \mathrm{dom}(D) < \min \mathrm{dom}(F)$.

(iii) $M \models$ "F is a sequence of measures".

Then the *set of measurables* in M is $\mathrm{meas}(M) := \mathrm{dom}(F \cap M)$.

The *low part* of M is: $\mathrm{lp}(M) := H_\kappa^M$, if $\kappa = \min \mathrm{meas}(M)$ exists, and $\mathrm{lp}(M) = M$ otherwise (H_κ is the class of sets of hereditary cardinality $<\kappa$).

It is easy to see that every $\kappa \in \mathrm{meas}(M)$ is regular within M.

2.2. Definition. Let $M = J_\alpha[F, D]$ be a premouse over D and $\kappa \in \mathrm{meas}(M)$. $\pi : M \to M'$ is called the *ultrapower of M at κ* if:

(i) M' is transitive.

(ii) π is Σ_1-elementary.

(iii) $M' = \{\pi(f)(\kappa) \mid f \in M, f : \kappa \to M\}$.

(iv) For $x \in M$, $x \subset \kappa$: $x \in F(\kappa) \leftrightarrow \kappa \in \pi(x)$.

There is at most one ultrapower of M at κ and if it exists it can be obtained by the usual factoring of $({}^\kappa M) \cap M$ modulo $F(\kappa)$. M' will be a premouse over D again, and so the operation may be iterated:

2.3. Definition. Let $M = J_\alpha[F, D]$ be a premouse over D. Let $I = \langle \kappa_i \mid i < \theta \rangle$ be an *index*, i.e., some sequence of ordinals. Then a system

$$\mathrm{It}(M, I) = \langle \langle M_i \mid i \leq \theta \rangle, \langle \pi_{ij} \mid i \leq j \leq \theta \rangle \rangle$$

is called *the iterated ultrapower of M by I* if:

(i) $M_0 = M$.

(ii) $\pi_{ij} : M_i \to M_j$, $\pi_{ii} = \mathrm{id} \upharpoonright M_i$, $\pi_{ik} = \pi_{jk} \circ \pi_{ij}$ $(i \leq j \leq k \leq \theta)$.

(iii) For each limit ordinal $k \leq \theta$, $\langle M_k, \langle \pi_{ik} \mid i \leq k \rangle \rangle$ is the transitive direct limit of $\langle \langle M_i \mid i < k \rangle, \langle \pi_{ij} \mid i \leq j < k \rangle \rangle$.

(iv) M_i is a premouse over D $(i \leq \theta)$.

(v) If $i<\theta$ and $\kappa_i \in \text{meas}(M_i)$, then $\pi_{i,i+1}: M_i \to M_{i+1}$ is the ultrapower of M_i at κ_i.

(vi) If $i<\theta$ and $\kappa_i \notin \text{meas}(M_i)$, then $\pi_{i,i+1} = \text{id} \restriction M_i$ and $M_{i+1} = M_i$.

We shall write M_I for M_θ and $\pi_I : M \to M_I$ for $\pi_{0\theta}: M \to M_\theta$. $\text{It}(M, I)$ is called an *iteration* $\geq\kappa$ if $\kappa_i \geq \kappa$ for all $i<\theta$.

2.4. Lemma. *Let* $\text{It}(M, I)$ *be an iteration* $\geq \kappa$ *of* M *by* I. *Then*:

(i) $\pi_I : M \to M_I$ *is* Σ_1*-elementary.*

(ii) $H_\kappa^M = H_\kappa^{M_I}$ *and* $\pi_I \restriction H_\kappa^M = \text{id} \restriction H_\kappa^M$ (*in particular,* $\pi_I \restriction \kappa = \text{id} \restriction \kappa$).

(iii) $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap M_I$.

2.5. Definition. A premouse M over D is *iterable* if $\text{It}(M, I)$ exists for all indices I.

We are now giving a few criteria for M to be iterable. Recall that a simple predicate F is *countably complete* if for every $\kappa \in \text{dom}(F)$ and $\{X_n \mid n<\omega\} \subset F(\kappa)$: $\bigcap \{X_n \mid n<\omega\} \neq \emptyset$.

2.6. Theorem. *Let* $M = J_\alpha[F, D]$ *be a premouse over* D.

(i) *If* F *is countably complete, then* M *is iterable.*

(ii) *Assume that* M *is iterable and that* $\pi : \bar{M} \to M$ *is* Σ_1*-elementary,* $\bar{M} = J_{\bar{\alpha}}[\bar{F}, \bar{D}]$. *Then* $\bar{M}$ *is an iterable premouse over* $\bar{D}$.

For (ii), one shows that every iterate of $\bar{M}$ can be embedded canonically in some iterate of M so that π, the iteration maps, and the embeddings commute.

2.7. Theorem. *Let* Q *be a transitive model of a sufficiently large finite part of* ZFC *and assume* $\omega_1 \subset Q$. *Then iterable premice are absolute for* Q, *i.e., if* $D, M \in Q$ *then*: M *is an iterable premouse over* D *iff* $Q \models$ "M *is an iterable premouse over* D".

2.8. Theorem. *Let* Q *be a transitive model of a sufficiently large part of* ZFC, *and let iterable premice be absolute for* Q. *Let* $j : \bar{Q} \to Q$ *be elementary, and* $\bar{Q}$ *transitive. Then iterable premice are absolute for* $\bar{Q}$, *too.*

The main tool in working with iterable premice is to 'compare' different ones by iterating them up to members of the same relativized J-hierarchy. This requires a coherence condition for the measure sequences to hold (Mitchell [11, 12]), and we shall ensure this condition by studying premice whose sequence of measurables is 'short':

2.9. Definition. Let $M = J_\alpha[F, D]$ be a premouse over D. M is called *short* if either (i) $D=\emptyset$ and opt(meas$(M)\cap\gamma)<$ min meas(M) for all $\gamma<\omega\alpha$ or (ii) $D\neq\emptyset$ and otp meas$(M)\leq$ min dom(D).

A *D-premouse* is a short premouse over D, and a *D-mouse* is an iterable short premouse over D.

The shortness of a D-mouse M implies that for any iteration $\pi_I:M\to M_I$ we have meas$(M_I)=\pi_I''$ meas(M); no 'new' measurables come up in the iteration which greatly simplifies the structural analysis of iterates and many 'iterate and compare'-arguments.

In the proof of the following theorem, both D-mice are iterated up to a simple predicate C consisting of closed unbounded filters.

2.10. Theorem. *Let M, N be D-mice. Then there are iterations $\pi_I:M\to M_I$, $\pi_J:N\to N_J$ such that M_I, N_J are of the form $M_I=J_\gamma[C,D]$, $N_J=J_\delta[C,D]$ for some simple predicate C. Assume further that $\pi_{I^*}:M\to M_{I^*}$, $\pi_{J^*}:N\to N_{J^*}$ is another pair of iterations to a common predicate $C^*:M_{I^*}=J_{\gamma^*}[C^*,D]$, $N_{J^*}=J_{\delta^*}[C^*,D]$. Then: $\gamma\leq\delta$ iff $\gamma^*\leq\delta^*$.*

2.11. Definition. For D-mice M, N set $M\leq_D^* N$ iff there are iterations $\pi_I:M\to M_I$, $\pi_J:N\to N_J$ such that $M_I=J_\gamma[C,D]$, $N_J=J_\delta[C,D]$, C is simple, and $\gamma\leq\delta$. Define $M\sim_D N$ iff $M\leq_D^* N$ and $N\leq_D^* M$.

Using 2.10, we get:

2.12. Theorem. (i) *$\sim_D$ is an equivalence relation on the class of D-mice.*

(ii) *$\leq_D^*$ induces a well-ordering of the $\sim_D$-equivalence classes.*

(iii) *If M, N are D-mice and $\sigma:M\to N$ is Σ_1-elementary then $M\leq_D^* N$.*

(iv) *If $M\sim_D N$ are D-mice, $M=J_\alpha[F,D]$, $N=J_\beta[G,D]$, and if $(F\restriction\gamma)\cap M\cap N=(G\restriction\gamma)\cap M\cap N$, then $\mathcal{P}(\gamma)\cap M=\mathcal{P}(\gamma)\cap N$ $(\gamma<\omega\alpha,\ \omega\beta)$.*

The set $0^{\#}\subset\omega$ defined by Solovay [19] can be viewed as the smallest mouse transcending all the J_α, $\alpha\in$ On (the existence of $0^{\#}$ is equivalent to the existence of an iterable premouse $J_{\kappa+1}[U]\models$"U is a measure on κ"). Likewise, one can define a real number $0^{\text{long}}\subset\omega$ coding the smallest iterable premouse over $\emptyset$ which is not short. Analogous to the Jensen [3] covering theorem for L which holds under the assumption "$0^{\#}$ does not exist", we will obtain a covering result assuming that "0^{long} does not exist". This is no restriction for the intended applications as demonstrated by Theorem 2.14. There is no need here to exhibit 0^{long} explicitely and we define:

2.13. Definition. "0^{long} exists" means that there is an iterable premouse over $\emptyset$ which is not short. "$\neg 0^{\text{long}}$" stands for "0^{long} does not exist".

2.14. Theorem. *If 0^{long} exists, then for every $\alpha \in \mathrm{On}$ there is an inner model with α measurable cardinals $<\alpha^+ + \omega_2$.*

To get this one iterates the non-short iterable premouse at its smallest measurable until the iterate has $\geq \alpha + 1$ measurables. The first α measures will then be measures in the associated Mitchell model.

3. Short core models

3.1. Definition. Let D be simple such that $D = \emptyset$ or $\operatorname{opt} \operatorname{dom}(D) \leq \min(D)$. Define the class $K[D]$ as:

$$K[D] := \bigcup \{\operatorname{lp}(M) \mid M \text{ is a } D\text{-mouse}\}.$$

For $\alpha \in \mathrm{On}$ set $K_\alpha[D] := H_\alpha^{K[D]}$.

Fix $K[D]$ for the moment.

3.2. Theorem. *$K[D]$ is a transitive inner model of* ZFC.
$\bar{D} := D \cap K[D] \in K[D]$ and $K[D] \models$ "$V = K[\bar{D}]$". If $\alpha > \sup \operatorname{dom}(D)$ is an uncountable cardinal in $K[D]$, then $K_\alpha[D] \models$ "$V = K[\bar{D}]$".

3.3. Lemma. *Let Q be a transitive model of a sufficiently large finite part of* ZFC + "$V = K[\bar{D}]$", *where $\bar{D} := D \cap Q \in Q$. Assume $\operatorname{dom}(\bar{D}) = \operatorname{dom}(D)$ and that $\bar{D}$-mice are absolute for Q. Then $Q \subset K_\alpha[D]$ where $\alpha = \mathrm{On} \cap Q \leq \infty$.*

3.4. Theorem. *For $x, y \in K[D]$ set $x \leq_D y$ iff $x \leq_M y$ for every D-mouse M where $x, y \in \operatorname{lp}(M)$. Then $\leq_D$ is a well-ordering of $K[D]$.*

This is due to the fact that $\leq_M$ is uniformly $\Sigma_1(M)$ and hence preserved under iterations of premice. The following technical result will be used in Section 6:

3.5. Lemma. *Let $\sigma : P \to Q$ be an elementary map between transitive models of a sufficiently large finite part of* ZFC. *Let $D_Q := D \cap Q \in Q$. Let $Q \models$ "$V = K[D_Q]$" and let D_Q-mice be absolute for Q. Assume $D_P := D \cap P \in P$, $D_Q = \sigma(D_P)$, and $\operatorname{dom}(D_P) = \operatorname{dom}(D_Q) = \operatorname{dom}(D)$. Then $x \leq_D \sigma(x)$ for all $x \in P$.*

3.6. Definition. A model $K[U]$ is called a (*short*) *core model* if $K[U] \models$ "U is a sequence of measures". A set U such that $K[U]$ is a core model is called *strong*.

Most properties of short core models we have proved need the assumption "0^{long} does not exist"; this condition will be included as ($\neg 0^{\text{long}}$) in the statement of theorems.

3.7. Theorem ($\neg 0^{\text{long}}$). *Let $K[U]$ be a core model. Then $K[U] \models$ GCH. Other L-like combinatorial properties also hold in $K[U]$.*

3.8. Definition. For strong U, U' set $U \leqslant_e U'$ if $U = U' \upharpoonright \eta$ for some $\eta \in \mathrm{On}$ ("U' is an *end-extension* of U"). Set $U <_e U'$ if $U \leqslant_e U'$ and $U \neq U'$. A strong U is *maximal* if there is no $U' >_e U$.

3.9. Theorem ($\neg 0^{\text{long}}$). *Let U be strong and $\gamma \in \mathrm{On}$. Then*:
- (i) *$U \upharpoonright \gamma$ is strong.*
- (ii) *$K[U \upharpoonright \gamma] \subset K[U]$, indeed $K[U \upharpoonright \gamma] = (K[\bar{U} \upharpoonright \gamma])^{K[U]}$ where $\bar{U} = U \cap K[U]$.*
- (iii) *$\mathcal{P}(\gamma) \cap K[U \upharpoonright \gamma] = \mathcal{P}(\gamma) \cap K[U]$.*

3.10. Theorem ($\neg 0^{\text{long}}$). *Let $\langle U_\beta \mid \beta < \eta \rangle$ be a $\leqslant_e$-ascending chain of strong sets. Then $\bigcup \{U_\beta \mid \beta < \eta\}$ is strong.*

Theorem 3.10 readily implies:

3.11. Theorem ($\neg 0^{\text{long}}$). *Let U be strong. Then there is a maximal strong $U' \geqslant_e U$.*

3.12. Lemma ($\neg 0^{\text{long}}$). *Let U be strong. Then for every regular cardinal η: $\sup(\mathrm{dom}(U) \cap \eta) < \eta$.*

3.13. Theorem ($\neg 0^{\text{long}}$). *Let U be strong and let $j: K[U] \to W$ be elementary, W transitive. Then $W = K[U']$ where $U' = j(U \cap K[U])$.*

The following results show that under $\neg 0^{\text{long}}$ the family of short core models is generated from a unqiue *canonical core model* by iterated ultrapowers. Iterated ultrapowers of inner models are very similar to iterated ultrapowers of premice (2.3).

3.14. Theorem ($\neg 0^{\text{long}}$). *Let $K[U]$, $K[U']$ be core models with $\mathrm{dom}(U) = \mathrm{dom}(U')$. Then $|K[U]| = |K[U']|$ and $U \cap K[U] = U' \cap K[U]$.*

3.15. Definition. Let U_{can} be the unique maximal strong sequence satisfying (i) $U_{\text{can}} \subset K[U_{\text{can}}]$, and (ii) if $\kappa \in \mathrm{dom}(U_{\text{can}})$, then κ is the minimal ordinal ζ such that there is some $U' >_e U_{\text{can}} \upharpoonright \kappa$ with $\zeta = \min \mathrm{dom}(U' \setminus (U_{\text{can}} \upharpoonright \kappa))$. U_{can} is called the *canonical sequence* and $K[U_{\text{can}}]$ is the *canonical core model.*

3.16. Theorem ($\neg 0^{\text{long}}$). *Let $K[U]$ be a core model. Then there is an iteration $\pi_I: K[U_{\text{can}}] \to K[U']$ such that $U' \geqslant_e U$.*

3.17. Theorem ($\neg 0^{\text{long}}$). *Let $j: K[U_{\text{can}}] \to K[U]$ be elementary. Then j is the* *iteration map of a normal iterated ultrapower of* $K[U_{\text{can}}]$, *normal meaning that its index is a strictly increasing sequence of ordinals.*

3.18. Theorem ($\neg 0^{\text{long}}$) **(Embedding Theorem).** *Let* $K[U]$ *be a core model and let* $j: K[U] \to K[U]$ *be elementary with critical point* $\kappa > \sup \operatorname{dom}(U)$. *Assume* δ *is a regular cardinal* $>\kappa$ *which is a limit cardinal in* $K[U]$. *Then there exists a strong* $U' >_e U$ *with* $\tau := \min \operatorname{dom}(U' \setminus U)$ *satisfying* (i) $\tau \geqslant \kappa$, *and* (ii) $\tau = \delta$ *if* $\delta = \omega_1$ *and* $\tau < \delta$ *if* $\delta \geqslant \omega_2$.

The following is a covering theorem for situations away from the measurables of the canonical sequence:

3.19. Theorem ($\neg 0^{\text{long}}$) **(Covering Theorem).** *Let* $\tau \geqslant \omega_2$ *such that* $\sup \operatorname{dom}(U_{\text{can}} \upharpoonright \tau + 1) < \tau$. *Let* $X \subset \tau$, $\operatorname{card}(X)$ *regular, and* $\operatorname{card}(X) < \operatorname{card}(\tau)$. *Then there exists* $Z \in K[U_{\text{can}}]$, $X \subset Z$ *such that* $\operatorname{card}^{K[U_{\text{can}}]}(Z) < \tau$.

3.20. Theorem ($\neg 0^{\text{long}}$). *Let* τ *be an ordinal with* $\sup \operatorname{dom}(U_{\text{can}} \upharpoonright \tau + 1) < \tau$. *Then*:

(i) *If* $\tau \geqslant \omega_2$ *is a limit ordinal and* $\operatorname{cof}(\tau) < \operatorname{card}(\tau)$, *then* τ *is singular in* $K[U_{\text{can}}]$.

(ii) *If* τ *is a singular cardinal in* V, *then* τ *is singular in* $K[U_{\text{can}}]$ *and* $\tau^+ = \tau^{+K[U_{\text{can}}]}$.

Proof. (i) Take $X \subset \tau$ cofinally such that $\operatorname{card}(X) = \operatorname{cof}(\tau) < \operatorname{card}(\tau)$. By 3.19 there is $Z \in K[U_{\text{can}}]$ such that $X \subset Z$ and $\operatorname{card}^{K[U_{\text{can}}]}(Z) < \tau$. So τ is singular in $K[U_{\text{can}}]$.

(ii) τ is singular in $K[U_{\text{can}}]$ by (i). Assume $\tau' := \tau^{+K[U_{\text{can}}]} < \tau^+$. Then $\operatorname{cof}(\tau') < \tau = \operatorname{card}(\tau')$. We can apply (i) to τ' and get τ' being singular in $K[U_{\text{can}}]$. Contradiction. $\square$

3.21. Theorem ($\neg 0^{\text{long}}$). *For all* $X \subset \tau \in \mathrm{On}$, $\operatorname{card}(X)$ *regular there exists* $Y \in K[U_{\text{can}}]$ *such that* $X \subset Y$, *and* $\operatorname{card}(Y) \leqslant \operatorname{card}(X) + \operatorname{card}(\gamma) + \omega_1$, *where* $\gamma = \sup \operatorname{dom}(U_{\text{can}} \upharpoonright \tau + 1)$.

Proof. Set $U := U_{\text{can}}$. Assume that $X \subset \tau$ is a counterexample to the theorem with τ minimal. Set $\gamma := \sup \operatorname{dom}(U_{\text{can}} \upharpoonright \tau + 1)$. Obviously $\tau > \gamma$, $\tau \geqslant \omega_2$, and $\operatorname{card}(X) < \operatorname{card}(\tau)$. By 3.19, there is $Z \in K[U]$ such that $X \subset Z$ and $\operatorname{card}^{K[U]}(Z) < \tau$. Take $\bar{\tau} < \tau$, $f \in K[U]$ so that $f : \bar{\tau} \leftrightarrow Z$. Set $\bar{X} := f^{-1}{}''X$. By the minimality of τ there is $\bar{Y} \in K[U]$, $\bar{X} \subset \bar{Y} \subset \bar{\tau}$ and $\operatorname{card}(\bar{Y}) \leqslant \operatorname{card}(X) + \operatorname{card}(\gamma) + \omega_1$. But then $Y := f''\bar{Y} \in K[U]$, $X \subset Y$, and $\operatorname{card}(Y) \leqslant \operatorname{card}(X) + \operatorname{card}(\gamma) + \omega_1$. Contradiction. $\square$

To obtain covering properties close to the measure sequence we have to extend $K[U_{\text{can}}]$ by a *Prikry system* (note that the universe could be a Prikry extension of some core model).

3.22. Definition. Let $K[U]$ be a core model. A function $C:\mathrm{dom}(U)\to V$ is called a *Prikry system* for $K[U]$ if:

(i) $C(\kappa)\subset\kappa$ and $\mathrm{otp}(C(\kappa))\le\omega$ for $\kappa\in\mathrm{dom}(U)$.

(ii) If $\langle x_\kappa \mid \kappa\in\mathrm{dom}(U)\rangle\in K[U]$ there is a finite set $p\subset\mathrm{On}$ such that

$$\forall\kappa\in\mathrm{dom}(U)\,(C(\kappa)\setminus p\ne\emptyset\to(x_\kappa\in U(\kappa)\leftrightarrow C(\kappa)\setminus p\subset x\subset\kappa)).$$

The extension of $K[U]$ by a Prikry system C is denoted by $K[U, C]$.

3.23. Theorem ($\neg 0^{\mathrm{long}}$) (Covering Theorem with Prikry sytem). *There is a Prikry system C for $K[U_{\mathrm{can}}]$ such that the following holds: Let $\tau\ge$* $(\mathrm{card}\,\mathrm{otp}\,\mathrm{dom}(U_{\mathrm{can}}))^{++}$. *Let $X\subset\tau$,* $\mathrm{card}(X)$ *regular, and* $\mathrm{card}(X)<\mathrm{card}(\tau)$. *Then there is a function $f:(\tau)^{<\omega}\to\tau$, $f\in K[U_{\mathrm{can}}]$ and a $\gamma<\tau$ such that*

$$X\subset Z:=\{f(\vec{\nu},\vec{\mu})\mid\vec{\nu}<\gamma,\ \vec{\mu}\in\bar{C}\cap\tau\},$$

where $\bar{C}:=\bigcup\{C(\kappa)\mid\kappa\in\mathrm{dom}(U)\}$. *In particular,* $\mathrm{card}^{K[U_{\mathrm{can}},C]}(Z)<\tau$.

The next result will be used as a condensation criterion in the two following sections:

3.24. Theorem. *Let D be a simple predicate with* $\mathrm{otp}\,\mathrm{dom}(D)\le\min(D)$ *or $D=\emptyset$. Let Q be a transitive model of a sufficiently large finite part of* ZFC + "$V=K[\bar{D}]$", $\bar{D}=D\cap Q\in Q$. *Let $\omega_1\subset Q$ and* $\mathrm{dom}(D)=\mathrm{dom}(\bar{D})$. *Then*:

(i) *Let M be a D-mouse,* $\mathrm{meas}(M)\ne\emptyset$ *and let $\kappa=$* $\min\mathrm{meas}(M)$ *be singular in Q. Then* $\mathrm{lp}(M)\subset Q$.

(ii) *Let $\lambda\subset Q$ be a cardinal* $>\sup(D)$ *and assume the following condition is satisfied*:

If $C\subset\lambda$ is closed unbounded in λ then there exists a $\kappa\in C$ which is singular in Q.

Then $K_\lambda[D]\subset Q$.

Proof. (i) Let $f\in Q$, $f:\nu\to\kappa$ cofinal, $\nu<\kappa$. $\bar{D}$-mice are absolute for Q (2.7), and we can take a D-mouse $N\in Q$ such that $f\in\mathrm{lp}(N)$. Let M^*, N^* be iterates of M, N respectively such that $N^*\subset M^*$ or $M^*\in N^*$ (2.10). If $N^*\subset M^*$ then $f\in\mathcal{P}(\kappa)\cap N=\mathcal{P}(\kappa)\cap N^*\subset\mathcal{P}(\kappa)\cap M^*=\mathcal{P}(\kappa)\cap M$, contradicting the regularity of κ inside M. Thus $M^*\in N^*$, and $\mathrm{lp}(M)=H^M_\kappa=H^{M^*}_\kappa\subset H^{N^*}_\kappa=H^N_\kappa\subset Q$.

(ii) Let $x\in K_\lambda[D]$. Take a D-mouse M such that $x\in\mathrm{lp}(M)$ and $\mathrm{card}(M)<\lambda$ (3.2). If $\mathrm{meas}(M)=\emptyset$, then $M\in L_\lambda[D]\subset Q$. So assume $\mathrm{meas}(M)\ne\emptyset$. Taking the ultrapower by the smallest measurable λ-times we obtain an iteration $\mathrm{It}(M,I)=\langle\langle M_i\mid i\le\lambda\rangle,\langle\pi_{ij}\mid i\le j\le\lambda\rangle\rangle$ of M by $I=\langle\kappa_i\mid i<\lambda\rangle$ so that $C:=\{\kappa_i\mid i<\lambda\}$ is closed unbounded in λ. By the condition, some κ_i, $i<\lambda$, is singular in Q. By (i), $x\in\mathrm{lp}(M)\subset\mathrm{lp}(M_i)\subset Q$. $\square$

In subsequent proofs, condensation arguments will yield embeddings of some initial segment of a core model. The final result in this section shows how to get an embedding of the entire core model from this.

3.25. Theorem ($\neg 0^{\text{long}}$). *Let $K[U]$ be a core model. Let λ be a cardinal $>\gamma:=\sup\operatorname{dom}(U)$, $\lambda\geqslant\omega_1$. Assume $\pi:K_\lambda[U]\to W$ is elementary, W transitive, and π has critical point $\alpha>\gamma$. Then there is an elementary embedding $\bar\pi:K[U]\to K[U]$ with critical point α.*

Proof. Set $G:=\{x\subset\alpha\mid x\in K_\lambda[U]\wedge\alpha\in\pi(x)\}$. By 3.13, it suffices to show that the ultrapower $\operatorname{Ult}(K[U],G)$ is well-founded. So assume not and let $f_0,f_1,\ldots\in K[U]$ be functions such that $\{\nu<\alpha\mid f_{n+1}(\nu)\in f_n(\nu)\}\in G$ for $n<\omega$. Let $\{f_n\mid n<\omega\}\subset K_\eta[U]$ where $K_\eta[U]$ reflects enough properties of $K[U]$. There is an elementary map $\sigma:Q\to K_\eta[U]$ so that Q is transitive, $\alpha\cup\{f_n\mid n<\omega\}\subset\operatorname{range}(\sigma)$, and $\operatorname{card}(Q)<\lambda$. Let $\bar U:=U\cap Q$; $\bar U\in Q$. By 2.8, $\bar U$-mice are absolute for Q, and by 3.3, $Q\subset K_\lambda[U]$. Set $\bar f_n:=\sigma^{-1}(f_n)$, $(n<\omega)$. Then, for $n<\omega$:

$$\{\nu<\alpha\mid\bar f_{n+1}(\nu)\in\bar f_n(\nu)\}=\{\nu<\alpha\mid f_{n+1}(\nu)\in f_n(\nu)\}\in G,$$

hence

$$\alpha\in\pi(\{\nu<\alpha\mid\bar f_{n+1}(\nu)\in\bar f_n(\nu)\})=\{\nu<\pi(\alpha)\mid\pi(\bar f_{n+1})(\nu)\in\pi(\bar f_n)(\nu)\},$$

and $\pi(\bar f_{n+1})(\alpha)\in\pi(\bar f_n)(\alpha)$. Contradiction. □

4. Chang's conjecture

4.1. Definition. For $\kappa,\lambda,\mu,\nu,\theta$ cardinals let $(\kappa,\lambda)\twoheadrightarrow_\theta(\mu,\nu)$ be the assertion:

> Every structure $\langle A,B,\ldots\rangle$ where $\operatorname{card}(A)=\kappa$, $\operatorname{card}(B)=\lambda$ and whose type has cardinality $\leqslant\theta$ possesses an elementary substructure $\langle\bar A,\bar B,\ldots\rangle$ such that $\operatorname{card}(\bar A)=\mu$, $\operatorname{card}(\bar B)=\nu$.

$(\kappa,\lambda)\twoheadrightarrow_\omega(\mu,\nu)$ is called *Chang's conjecture* for the pairs (κ,λ), (μ,ν).

Chang's conjecture is presented in Chang–Keisler [1, 7.3.1], and various relations between instances of Chang's conjecture are discussed there.

4.2. Lemma. $(\mu^{++},\mu^+)\twoheadrightarrow_\omega(\mu^+,\mu)$ *implies* $(\mu^{++},\mu^+)\twoheadrightarrow_\mu(\mu^+,\mu)$.

Proof. Assume $(\mu^{++},\mu^+)\twoheadrightarrow_\omega(\mu^+,\mu)$. Consider a structure $S=\langle\mu^{++},\mu^+,\langle R_\nu\mid\nu<\mu\rangle,\langle f_\nu\mid\nu<\mu\rangle\rangle$ with relations R_ν and functions f_ν. We suppose that the family of f_ν's contains a set of Skolem functions for S. Define a function F: $(\mu^{++})^{<\omega}\to\mu^+$ by:

$$F(\vec x)=\sup(\{f_\nu(\vec x):\nu<\mu\}\cap\mu^+)<\mu^+.$$

By $(\mu^{++}, \mu^+) \Rightarrow_\omega (\mu^+, \mu)$ there is $X \prec \langle \mu^{++}, \mu^+, F \rangle$ such that $\mathrm{card}(X) = \mu^+$ and $\mathrm{card}(X \cap \mu^+) = \mu$. Set $Y := \bigcup \{f_\nu(\vec{x}) \mid \nu < \mu,\ \vec{x} \in X\}$. $Y \prec S$, $\mathrm{card}(Y) = \mu^+$, and $Y \cap \mu^+ \subset \sup(X \cap \mu^+) < \mu^+$. □

4.3. Theorem. *Assume* $(\mu^{++}, \mu^+) \Rightarrow_\omega (\mu^+, \mu)$, $\mu \geq \omega_1$. *Then* 0^{long} *exists.*

Proof. Assume $\neg 0^{\mathrm{long}}$. Let $U := U_{\mathrm{can}} \restriction \mu^{++}$. By 3.12,

(1) $\sup \mathrm{dom}(U \restriction \mu^+) < \mu^+$ and $\sup \mathrm{dom}(U) < \mu^{++}$.

Let H be a transitive structure reflecting enough properties of V with $\mu^{++}, U \in H$. By $(\mu^{++}, \mu^+) \Rightarrow_\mu (\mu^+, \mu)$ there is an $X \prec H$ such that $\mu^+, \mu^{++}, U \in X$, $(\mu + 1) \subset X$, $\mathrm{card}(X \cap \mu^{++}) = \mu^+$, and $\mathrm{card}(X \cap \mu^+) = \mu$. Let $\pi : \bar{H} \cong X \prec H$, $\bar{H}$ transitive. Set $\alpha := \pi^{-1}(\mu^+)$, $\beta := \pi^{-1}(\mu^{++})$, $\bar{U} := \pi^{-1}(U)$.

(2) $X \cap \mu^+$ is transitive.

Proof. Let $\gamma \in X \cap \mu^+$. There is $f \in X$, $f : \mu \to \gamma$ onto. Since $\mu \subset X$, $\gamma = f''\mu \subset X$. □(2)

Thus:

(3) $\alpha = X \cap \mu^+$ is the critical point of π.

(4) $\sup \mathrm{dom}(U \restriction \mu^+) < \alpha$, since $\sup \mathrm{dom}(U \restriction \mu^+) \in X$.

(5) $\beta = \mu^+$.

Proof. $\beta \geq \mu^+$ since $\mathrm{card}(X \cap \mu^{++}) = \mu^+$. If $\beta > \mu^+$, then $\pi(\mu^+)$ is a cardinal in V, and $\mu^+ = \pi(\alpha) < \pi(\mu^+) < \pi(\beta) = \mu^{++}$. Contradiction. □(5)

Set $Q := (K[\bar{U}])^{\bar{H}}$. By (1), $\sup \mathrm{dom}(\bar{U}) < \beta$.

(6) $K_\beta[\bar{U}] \subset Q$.

Proof. We apply 3.24(ii). Let $C \subset \beta$ be closed unbounded in β. Choose $\kappa \in C$ such that $\kappa > \sup \mathrm{dom}(\bar{U})$, α, and $\mathrm{cof}(\kappa) \neq \mathrm{cof}(\alpha)$; this is possible since there are at least two different cofinalities below $\mu^+ \geq \omega_2$. $\bar{H} \vDash \mathrm{cof}(\kappa) \neq \mathrm{cof}(\alpha)$, hence $\mathrm{cof}(\pi(\kappa)) \neq \mathrm{cof}(\pi(\alpha)) = \mu^+$. $\mathrm{cof}(\pi(\kappa)) < \mu^+ = \mathrm{card}(\pi(\kappa))$ and $\pi(\kappa) > \sup \mathrm{dom}(U)$. By 3.20(i), $K[U] \vDash$ "$\pi(\kappa)$ is singular". Thus $Q \vDash$ "κ is singular", as required. □(6)

(7) $K[\bar{U}]$ is a core model.

Proof. $K[\bar{U}] \vDash$ "$\bar{U}$ is a sequence of measures", since $K_\beta[\bar{U}] \subset Q$ and $Q \vDash$ "$\bar{U}$ is a sequence of measures". □(7)

$\mathrm{dom}(\bar{U} \restriction \alpha) = \mathrm{dom}(U \restriction \alpha)$, and so by 3.14, $\bar{U} \restriction \alpha = U \restriction \alpha$. By (4), there is no end-extension of $U \restriction \alpha$ with new measurables below $\mu^+ = \beta$, and so $\bar{U} = \bar{U} \restriction \alpha = U \restriction \beta$. (6) and some absoluteness considerations imply:

(8) $K_\beta[U]=(K_\beta[\bar{U}])^{\bar{H}}$.

So $\pi\restriction K_\beta[U]:K_{\mu^+}[U]\to K_{\mu^{++}}[U]$ is elementary with critical point α. By 3.25 there is $\bar{\pi}$ such that

(9) $\bar{\pi}:K[U\restriction\mu^+]\to K[U\restriction\mu^+]$ is elementary with critical point α.

(10) μ^+ is a limit cardinal in $K[U\restriction\mu^+]$.

Proof. Assume instead $\mu^+=\eta^{+K[U\restriction\mu^+]}$, $\eta\in\mathrm{card}^{K[U\restriction\mu^+]}$. $\eta\in X\cap\mu^+=\alpha<\mu^+$ implies $K[U\restriction\mu^+]\models$ "α is singular", which contradicts (9). $\square$(10)

By the embedding Theorem 3.18 there exists $U'>_e U\restriction\mu^+$ with $\min\mathrm{dom}(U'\setminus(U\restriction\mu^+))<\mu^+$. But this contradicts the definition of U_{can}. $\square$

5. 'Accessible' Jónsson cardinals

5.1. Definition. A cardinal λ is called a *Jónsson cardinal* provided every structure of cardinality λ whose type is countable possesses a proper elementary substructure of cardinality λ.

For details on Jónsson cardinals see Chang–Keisler [1, 7.3.2]. Every Jónsson cardinal is $\geqslant\omega_\omega$ [1, exercise 7.3.15].

5.2. Theorem. *Let λ be a Jónsson cardinal and $\lambda=\omega_\xi$, $\xi<\lambda$. Then 0^{long} exists.*

Proof. We assume $\neg 0^{\mathrm{long}}$ and work for a contradiction.

Let $U:=U_{\mathrm{can}}$. Let H be a transitive structure reflecting enough properties of the universe with λ, $U\in H$. Since λ is Jónsson there is $X\prec H$ such that λ, $U\in X$, $\mathrm{card}(X\cap\lambda)=\lambda$ and $X\cap\lambda\neq\lambda$. Let $\pi:\bar{H}\cong X\prec H$, $\bar{H}$ transitive. Set $\bar{U}:=\pi^{-1}(U)$. Let $\alpha<\lambda$ be the critical point of π; let $\alpha':=\pi(\alpha)$.

(1) $[\alpha,\alpha')\cap X=\emptyset$.

(2) α' is regular in V, $\alpha'\geqslant\omega_1$.

Proof. If $\mathrm{cof}(\alpha')<\alpha$, there is a sequence in X converging to α', contradicting (1). If $\alpha\leqslant\mathrm{cof}(\alpha')<\alpha'$, then $\mathrm{cof}(\alpha')\in X$ which again contradicts (1). $\square$(2)

(3) $\sup\mathrm{dom}(U\restriction\alpha')<\alpha$ and $\mathrm{dom}(U\restriction\alpha')=\mathrm{dom}(\bar{U}\restriction\alpha)$.

$Z:=\{\beta<\lambda\mid\bar{H}\models$ "β is a cardinal"$\}$ is a closed set of ordinals of ordertype $\leqslant\xi<\lambda$. Choose a regular cardinal μ such that ω_2, α, $\xi<\mu\leqslant\lambda$. Set $\mu':=\pi(\mu)$. Let $\theta:=\max(Z\cap\mu)<\mu$. Set $Q:=(K[\bar{U}\restriction\mu])^{\bar{H}}$. $\sup\mathrm{dom}(\bar{U}\restriction\mu)<\mu$, by 3.12.

(4) $K_\mu[\bar{U}\restriction\mu]\subset Q$.

Proof. We apply 3.24(ii). Let $C\subset\mu$ be closed unbounded in μ. Choose $\kappa\in C$

such that $\kappa > \sup \operatorname{dom}(\bar{U} \restriction \mu)$, θ, and $\operatorname{cof}(\kappa) \neq \operatorname{cof}(\theta)$. $\bar{H} \models \operatorname{cof}(\kappa) \neq \operatorname{cof}(\theta)$, and $\operatorname{cof}(\pi(\kappa)) \neq \operatorname{cof}(\pi(\theta))$. So $\operatorname{cof}(\pi(\kappa)) < \pi(\theta) = \operatorname{card}(\pi(\kappa))$, and $\pi(\kappa) > \sup \operatorname{dom}(U \restriction \mu')$. By 3.20(i), $K[U \restriction \mu'] \models$ "$\pi(\kappa)$ is singular", and thus $Q \models$ "κ is singular", as required. □(4)

(5) $\bar{U} \restriction \mu$ is strong.

Proof. $K[\bar{U} \restriction \mu] \models$ "$\bar{U} \restriction \mu$ is a sequence of measures", since $K_\mu[\bar{U} \restriction \mu] \subset Q$, and $Q \models$ "$\bar{U} \restriction \mu$ is a sequence of measures". □(5)

Since α is the critical point of π, $\operatorname{dom}(\bar{U} \restriction \alpha) = \operatorname{dom}(U \restriction \alpha)$. So by 3.14,

(6) $\bar{U} \restriction \alpha = U \restriction \alpha$.

By (3), $U \restriction \alpha = U \restriction \alpha'$. Since U is 'canonical':

(7) $[\alpha, \alpha') \cap \operatorname{dom}(\bar{U}) = \emptyset$.

(8) $\alpha' \notin \operatorname{dom}(U)$, since $\alpha \notin \operatorname{dom}(\bar{U})$.

Absoluteness considerations and (4) imply:

(9) $K_\mu[U \restriction \alpha] = (K_\mu[\bar{U} \restriction \alpha])^{\bar{H}}$.

The map $\pi \restriction K_\mu[U \restriction \alpha]: K_\mu[U \restriction \alpha] \to K_{\mu'}[U \restriction \alpha']$ is elementary with critical point α. By 3.25 there is $\tilde{\pi}$ such that:

(10) $\tilde{\pi}: K[U \restriction \alpha] \to K[U \restriction \alpha]$ is elementary with critical point α.

As in 4.3(10),

(11) $\alpha' = \pi(\alpha)$ is a limit cardinal in $K[U \restriction \alpha]$.

By the embedding Theorem 3.18 there exists a strong predicate $U' >_e U \restriction \alpha$ with $\min \operatorname{dom}(U' \setminus (U \restriction \alpha')) \leq \alpha'$. But by (3) and (8) this contradicts the fact that U is canonical. □

5.3. Theorem ($\neg 0^{\text{long}}$). *Let λ be a Jónsson cardinal such that one of the following holds*:

(i) $\omega < \operatorname{cof}(\lambda) < \lambda$.

(ii) *λ is regular but not weakly hyper-Mahlo.*

Then $\operatorname{dom}(U_{\text{can}} \restriction \lambda)$ *is cofinal in λ.*

A definition of *weakly hyper-Mahlo* can be found in Drake [8, Ch. 4, §3.6]. The subsequent proof also contains a definition of this notion.

Proof. Set $U := U_{\text{can}}$, and assume $\sup \operatorname{dom}(U \restriction \lambda) < \lambda$. By the previous theorem, $\lambda = \omega_\lambda$. Hence λ is a limit cardinal. Let H be a transitive structure reflecting enough properties of the universe with $\lambda, U \in H$. Since λ is Jónsson, there is $X \prec H$ such that $\lambda, U \in X$, $\operatorname{card}(X \cap \lambda) = \lambda$, and $X \cap \lambda \neq \lambda$. Let $\pi: \bar{H} \cong X \prec H$, $\bar{H}$

transitive. Set $\bar{U} := \pi^{-1}(U)$. Let $\alpha < \lambda$ be the critical point of π; $\alpha' := \pi(\alpha)$. As in the proof of 5.2 we get:

(1) α' is regular in V, $\alpha' \geq \omega_1$.

(2) $\sup \mathrm{dom}(U \restriction \alpha') < \alpha$.

Set $Q := (K[\bar{U} \restriction \lambda])^{\bar{H}}$. $\sup \mathrm{dom}(\bar{U} \restriction \lambda) < \lambda$.

(3) $K_\lambda[\bar{U} \restriction \lambda] \subset Q$.

Proof. We apply 3.24(ii). Let $C \subset \lambda$ be closed unbounded in λ.

Case (i): $\omega < \mathrm{cof}(\lambda) < \lambda$. Then there is a closed unbounded set $D \subset \lambda$ consisting of singular cardinals. We may assume that $D \in X$. $\bar{D} := \pi^{-1}(D)$ is closed unbounded in λ. Take $\kappa \in C \cap \bar{D}$ such that $\kappa > \sup \mathrm{dom}(\bar{U} \restriction \lambda)$. $\pi(\kappa)$ is a singular cardinal $> \sup \mathrm{dom}(U \restriction \lambda)$, and by 3.20(ii), $K[U] \models$ "$\pi(\kappa)$ is singular". Thus $Q \models$ "κ is singular", as required.

Case (ii): *λ is regular but not weakly hyper-Mahlo.* λ is weakly inaccessible since $\lambda = \omega_\lambda$. Adjoin a suitable set '-1' as a new least element to the ordinals. For $\beta \in \mathrm{On}$ define its (weak) *Mahlo degree* $M(\beta) \in [-1, \beta]$ by:

$M(\beta) \geq 0$ iff β is weakly inaccessible;

$M(\beta) \geq \gamma$ iff for all $\delta < \gamma$ the set $\{\eta < \beta \mid M(\eta) \geq \delta\}$ is stationary in β, $(\gamma > 0)$.

β is *weakly Mahlo* if $M(\beta) \geq 1$. β is *weakly Mahlo of kind* γ if $M(\beta) \geq \gamma \geq 0$. β is *weakly hyper-Mahlo* if $M(\beta) = \beta$. Thus $0 \leq M(\lambda) < \lambda$.

To every $\beta \leq \lambda$ with $0 \leq M(\beta) < \beta$ assign a closed unbounded set $D_\beta \subset \beta$ such that $\delta \in D_\beta$ implies δ is a limit cardinal and $M(\delta) < M(\beta)$. We may assume that the function $\langle D_\beta \mid \beta \leq \lambda,\ 0 \leq M(\beta) < \beta \rangle$ is an element of X.

Let $\bar{M}$ be defined in $\bar{H}$ as M is in V: $\bar{M} = (M)^{\bar{H}}$. For $\beta \leq \lambda$, $0 \leq \bar{M}(\beta) < \beta$ set $\bar{D}_\beta := \pi^{-1}(D_{\pi(\beta)})$. $\bar{D}_\beta$ is closed unbounded in β; $\delta \in \bar{D}_\beta \to \pi(\delta)$ is a limit cardinal and $\bar{M}(\delta) < \bar{M}(\beta)$.

We assume that the closed unbounded set $C \subset \lambda$ has $\min(C) > \sup \mathrm{dom}(\bar{U} \restriction \lambda)$. Do the following construction until the recursion breaks down:

Set $\beta_0 := \lambda$, $\gamma_0 := \bar{M}(\lambda) < \lambda$. If β_n, γ_n are defined, put $\beta_{n+1} :=$ the ω_{γ_n+1}-st element of $C \cap \bar{D}_{\beta_n}$, and $\gamma_{n+1} := \bar{M}(\beta_{n+1})$.

Claim. Let β_n, γ_n be constructed and assume $\mathrm{cof}(\beta_n) > \omega_{\gamma_n+1}$, $\bar{M}(\beta_n) \geq 0$. Then $\beta_{n+1}, \gamma_{n+1}$ exist and $\mathrm{cof}(\beta_{n+1}) > \omega_{\gamma_{n+1}+1}$. Also $\gamma_{n+1} < \gamma_n$.

Proof. Because $\gamma_n = \bar{M}(\beta_n) \geq 0$, $\bar{D}_{\beta_n}$ is closed unbounded in β_n. $\mathrm{cof}(\beta_n) > \omega_{\gamma_n+1} \geq \omega_1$, and $\beta_{n+1} =$ the ω_{γ_n+1}-st element of $C \cap \bar{D}_{\beta_n}$ exists. $\gamma_{n+1} = \bar{M}(\beta_{n+1}) < \bar{M}(\beta_n) < \gamma_n$, and so $\mathrm{cof}(\beta_{n+1}) = \omega_{\gamma_n+1} > \omega_{\gamma_{n+1}+1}$. □(Claim)

So β_1, γ_1 exist. Since $\gamma_n > \gamma_{n+1}$, the construction breaks down, and by the claim there must be $n \geq 1$ such that $\gamma_n = \bar{M}(\beta_n) = -1$. Set $\kappa := \beta_n$. Then $M(\pi(\kappa)) = -1$, and $\pi(\kappa)$ is a singular cardinal $> \sup \mathrm{dom}(U \restriction \lambda)$. By 3.20(ii), $K[U \restriction \lambda] \models$ "$\pi(\kappa)$ is singular". So $Q \models$ "κ is singular", as required. □(3)

The proof can now be finished exactly as the proof of 5.2 from 5.2(5) onwards, taking $\mu := \mu' := \lambda$. □

As corollaries we get the following results:

5.4. Theorem. *If there exists a regular Jónsson cardinal λ which is not weakly hyper-Mahlo, then 0^{long} exists.*

Proof. Assume $\neg 0^{\text{long}}$. By 5.3, $\text{dom}(U_{\text{can}} \restriction \lambda)$ is cofinal in λ, which contradicts 3.12. □

5.5. Theorem. *Let λ be a Jónsson cardinal with $\omega < \delta := \text{cof}(\lambda) < \lambda$. Then there is an inner model in which the set of measurables $<\lambda$ has ordertype $\geqslant \delta$.*

Proof. If 0^{long} exists this follows from 2.14. Assume that 0^{long} does not exist, on the other hand. By 5.3, $\text{dom}(U_{\text{can}} \restriction \lambda)$ is cofinal in λ, thus $\text{otp}\,\text{dom}(U_{\text{can}} \restriction \lambda) \geqslant \delta$. □

Conversely, we have the well-known result of Prikry [15]:

5.6. Theorem. *If $\langle \kappa_i \mid i < \delta \rangle$ is a strictly increasing sequence of measurable cardinals where δ is a limit ordinal $<\kappa_0$, then $\lambda := \sup\{\kappa_i \mid i < \delta\}$ is a Jónsson cardinal.*

Theorems 5.5 and 5.6 yield a family of equiconsistency results of which we present just one typical example:

5.7. Theorem. *The theories* "ZFC + *there is a Jónsson cardinal of cofinality ω_1*" *and* "ZFC + *there are ω_1 measurable cardinals*" *are equiconsistent.*

6. Free subsets

6.1. Definition. A subset X of a structure S is *free* in S if $\forall x \in X\, x \notin S[X \setminus \{x\}]$; here $S[Y]$ denotes the substructure of S generated from Y by the functions and constants of S. For cardinals κ, λ, μ let $\text{Fr}_\mu(\kappa, \lambda)$ be the assertion: Every structure S with $\kappa \subset S$ whose type has cardinality $\leqslant \mu$ possesses a free subset $X \subset \kappa$ of cardinality $\geqslant \lambda$.

$\text{Fr}_{<\kappa}(\kappa, \lambda)$ stands for: $\forall \mu < \kappa\, \text{Fr}_\mu(\kappa, \lambda)$.

This section extends techniques of [10] where we proved that $\text{Fr}_\omega(\omega_\omega, \omega)$ is equiconsistent to "there exists a measurable cardinal". The following lemma is [10, 1.1]:

6.2. Lemma. *Let λ be an infinite cardinal and assume $\mathrm{Fr}_\mu(\kappa, \lambda)$. Let S be a structure with $\kappa \subset S$ whose type has cardinality $\leq \mu$. Then there is a subset $X \subset \kappa$ free in S with monotone enumeration $\langle x_i \mid i < \lambda \rangle$ such that:*

(i) $i < \lambda \rightarrow [x_i, x_i^+) \cap S[x_i \cup \{x_j \mid i < j < \lambda\}] = \emptyset$

(x_i^+ is the smallest cardinal $> x_i$). In particular:

(ii) $i < \lambda \rightarrow x_i \notin S[x_i \cup \{x_j \mid i < j < \lambda\}]$.

6.3. Lemma. *Let λ, μ be infinite cardinals and let κ be the least cardinal such that $\mathrm{Fr}_\mu(\kappa, \lambda)$. Then:*

(i) *κ is a limit cardinal.*

(ii) $\mathrm{Fr}_{<\kappa}(\kappa, \lambda)$.

(iii) *κ is weakly inaccessible or* $\mathrm{cof}(\kappa) = \mathrm{cof}(\lambda)$.

(iv) *If $\mu = \omega_\xi$, then $\kappa \geq \omega_{\xi+\lambda}$.*

Proof. Just as Lemma 1.2 of [10]. $\square$

6.4. Theorem ($\neg 0^{\text{long}}$). *Let $\kappa = \omega_\xi$, $\xi < \kappa$ and* $\mathrm{Fr}_{<\kappa}(\kappa, \lambda)$, *where λ is an uncountable cardinal. Then* $\mathrm{dom}(U_{\mathrm{can}} \restriction \kappa)$ *is cofinal in κ.*

Proof. Set $U := U_{\mathrm{can}} \restriction \kappa$. Assume that $\nu := \sup \mathrm{dom}(U) < \kappa$. By 6.3(iv), κ is a limit cardinal and so $\mu := \max(\nu^+, \xi^+, \omega_2) < \kappa$.

(1) There is $\bar\kappa < \kappa$ so that $\mathrm{Fr}_\mu(\bar\kappa, \omega)$.

Proof. If there is $\bar\kappa < \kappa$ such that $\mathrm{Fr}_\mu(\bar\kappa, \lambda)$ we are done. So assume κ is minimal with $\mathrm{Fr}_\mu(\kappa, \lambda)$. By 6.3(iii), κ is singular and $\mathrm{cof}(\kappa) = \mathrm{cof}(\lambda)$. Take a cardinal $\bar\lambda < \lambda$ such that $\mathrm{cof}(\bar\lambda) \neq \mathrm{cof}(\lambda)$. By 6.3(iii), there has to be $\bar\kappa < \kappa$ such that $\mathrm{Fr}_\mu(\bar\kappa, \bar\lambda)$. Then $\mathrm{Fr}_\mu(\bar\kappa, \omega)$. $\square$(1)

Fix $\bar\kappa = \omega_\zeta < \kappa$ such that $\mathrm{Fr}_\mu(\bar\kappa, \omega)$. By the covering theorem 3.21 there exists $E \in K[U]$ such that $\{\omega_i \mid i < \zeta\} \subset E \subset \bar\kappa$ and $\mathrm{card}(E) < \mu$. Let $K_\theta[U]$, $\theta > \bar\kappa$ reflect sufficiently many properties of $K[U]$. Let $S := \langle K_\theta[U], E, \langle \alpha \mid \alpha \leq \mu \rangle \rangle$ augmented by a countable set of Skolem functions for $K_\theta[U]$; E and every $\alpha \leq \mu$ are understood to be constants of S. By 6.2(i), there is a set $X \subset \bar\kappa$ free in S with monotone enumeration $\langle x_i \mid i < \omega \rangle$ so that:

(2) $[x_i, x_i^+) \cap S[x_i \cup \{x_j \mid i < j < \omega\}] = \emptyset$ for all $i < \omega$.

For $i < \omega$ set $M_i := S[\{x_j \mid i \leq j < \omega\}]$ and let $\pi_i : M_i \cong \bar M_i$, $\bar M_i$ transitive. Set $U_i := \pi_i(U)$. For $i \leq j < \omega$ let $\pi_{ji} := \pi_i \circ \pi_j^{-1} : \bar M_j \rightarrow \bar M_i$.

(3) $\mu \subset M_i$ and $\pi_i \restriction \mu = \mathrm{id}$.

(4) $\bar M_i \models$ "$V = K[U_i]$", $U_i = U \cap \bar M_i \in \bar M_i$, and U_i-mice are absolute for $\bar M_i$ (the latter follows from 2.8).

(5) π_{ji} is elementary, $\pi_{ji} \upharpoonright \mu = \mathrm{id}$, and $\pi_{ji}(U_j) = U_i$.

For $i < \omega$ set $E_i := \pi_i(E)$; $\pi_{ji}(E_j) = E_i$. Then 3.5 implies:

(6) $E_j \leqslant_U E_i$, for $i \leqslant j < \omega$.

Since $\leqslant_U$ well-orders $K[U]$ (3.4) there is $i < \omega$ such that $E_{i+1} = E_i$. "x is the α-th element of E_i", for $\alpha < \mathrm{otp}(E_i) < \mu$, is uniformly definable in M_i and M_{i+1}, and since $\pi_{i+1,i} \upharpoonright \mu = \mathrm{id}$:

(7) $\pi_{i+1,i} \upharpoonright E_{i+1} = \mathrm{id}$.

Let $\delta := \pi_{i+1}(x_i^+) \in E_{i+1}$. Then:

$$\begin{aligned}\pi_{i+1,i}(\delta) = \pi_i(x_i^+) > \pi_i(x_i) &= \mathrm{otp}(M_i \cap x_i) \geqslant \mathrm{otp}(M_{i+1} \cap x_i) \\ &= \mathrm{otp}(M_{i+1} \cap x_i^+) \text{ (by (2))} = \pi_{i+1}(x_i^+) = \delta,\end{aligned}$$

which contradicts (7). □

6.5. Theorem. *Assume* $\mathrm{Fr}_\omega(\omega_\lambda, \lambda)$, *where λ is a cardinal with $\omega_1 \leqslant \lambda < \omega_\lambda$. Then there is an inner model in which the set of measurables $<\omega_\lambda$ has ordertype $\geqslant \lambda$.*

Proof. If 0^{long} exists the theorem follows by 2.14. So assume $\neg 0^{\text{long}}$. Set $\kappa := \omega_\lambda$. By 6.3, $\mathrm{Fr}_{<\kappa}(\kappa, \lambda)$ holds. We shall show that $\mathrm{otp}\,\mathrm{dom}(U_{\mathrm{can}} \upharpoonright \kappa) \geqslant \lambda$. By 6.4, $\mathrm{dom}(U_{\mathrm{can}} \upharpoonright \kappa)$ is cofinal in κ, and for regular λ this implies $\mathrm{otp}\,\mathrm{dom}(U_{\mathrm{can}} \upharpoonright \kappa) \geqslant \lambda$.

Now consider the case that λ is a singular cardinal. It suffices to show that $\mathrm{otp}(U_{\mathrm{can}} \upharpoonright \kappa) \geqslant \lambda'$ for every regular cardinal $\lambda' < \lambda$. Let $\lambda' < \lambda$ be regular, $\lambda' \geqslant \omega_1$. Let $\kappa' \leqslant \kappa$ be minimal such that $\mathrm{Fr}_\lambda(\kappa', \lambda')$; let $\kappa' = \omega_\zeta$. Then:

(1) $\zeta \leqslant \lambda < \kappa' = \omega_\zeta$.

(2) $\mathrm{Fr}_{<\kappa'}(\kappa', \lambda')$, by 6.3(ii).

(3) $\mathrm{cof}(\kappa') \geqslant \lambda'$, by 6.3(iii).

By 6.4, $\mathrm{dom}(U_{\mathrm{can}} \upharpoonright \kappa')$ is cofinal in κ'. Then

$$\mathrm{otp}\,\mathrm{dom}(U_{\mathrm{can}} \upharpoonright \kappa) \geqslant \mathrm{otp}\,\mathrm{dom}(U_{\mathrm{can}} \upharpoonright \kappa') \geqslant \mathrm{cof}(\kappa') \geqslant \lambda'. \quad \square$$

Conversely, Shelah [17] proves:

6.6. Theorem. *Assume* GCH. *Let $\langle \kappa_i \mid i < \lambda \rangle$ be a strictly increasing sequence of measurable cardinals, $\lambda = \omega_\xi$, $\xi < \lambda < \kappa_0$. Let $\kappa := \sup\{\kappa_i \mid i < \lambda\}$. Then there is a generic extension $V[G]$ of V satisfying*:

$$V[G] \models \text{“}\kappa = \omega_\lambda \text{ and } \mathrm{Fr}_\omega(\omega_\lambda, \lambda)\text{”}.$$

Theorem 6.5 and 6.6 imply a series of equiconsistency results of which we present one typical example:

6.7. Theorem. *The theories* "ZFC + $\mathrm{Fr}_\omega(\omega_{\omega_1}, \omega_1)$" *and* "ZFC + *there are* ω_1 *measurable cardinals*" *are equiconsistent.*

7. Canonical forms

7.1. Definition. Let the sequence $\langle \omega_{k(n)} \mid n < \omega \rangle$ be strictly increasing with supremum ω_ω. By CF(k) we denote the following combinatorial property:

For every sequence $\langle f_i \mid i < \omega \rangle$ of functions, $f_i : (\omega_\omega)^{2i} \to 2$ there exist sets $B_n \subset \omega_{k(n)} \setminus \omega_{k(n-1)}$, $\mathrm{card}(B_n) \geq 3$ such that for every $i < \omega$, $n_1 < \cdots < n_i < \omega$, and $x_1, y_1, x_1', y_1' \in B_{n_1}, \ldots, x_i, y_i, x_i', y_i' \in B_{n_i}$, with $x_1 < y_1 < x_2 < y_2 < \cdots < x_i < y_i$, $x_1' < y_1' < x_2' < y_2' < \cdots < x_i' < y_i'$ we have:

$$f_i(x_1, y_1, x_2, y_2, \ldots, x_i, y_i) = f_i(x_1', y_1', x_2', y_2', \ldots, x_i', y_i').$$

Such a sequence $\langle B_n \mid n < \omega \rangle$ will be called *homogeneous* for $\langle f_i \mid i < \omega \rangle$. The B_n's can be viewed as segments of a system of indiscernibles for the functions $\langle f_i \mid i < \omega \rangle$, so that the value of the functions is independent of which pair from a segment is chosen.

In the language of Shelah [16], who studies strong partition-properties of ω_ω, CF(k) is the same as: $\langle \omega_{k(n)} \mid n < \omega \rangle$ has a $\langle 3 \mid n < \omega \rangle$-*canonical form* for

$$\{\langle \underbrace{2, 2, \ldots, 2}_{i\text{-times}} \rangle_2^{2i} \mid i < \omega\}.$$

The consistency strengths of (reasonable) canonical form properties for ω_ω where just *singletons* from each segment of indiscernibles are considered correspond to the existence of one measurable cardinal (see [17] and [10]). So CF(k) is basically the next stage as far as the strengths of canonical forms are concerned. A model for CF(k)—indeed for a much stronger canonical form property—was constructed by Shelah [16] from a ground model with ω strongly compact cardinals (with $k(n) = (n+5) \cdot n/2 + n + 1$). The following theorem shows that high levels of measurability are indeed necessary for such a construction.

7.2. Theorem. *Assume* CF(k) *for some function* k. *Then* 0^{long} *exists.*

Proof. We assume $\neg 0^{\mathrm{long}}$ and work for a contradiction.

Set $\kappa := \omega_\omega$ and $U := U_{\mathrm{can}}$. Let H be a transitive structure reflecting enough properties of the universe and let $\kappa, U \in H$. Let H also possess a countable collection of Skolem functions for itself. We assume that the Skolem functions are suitably encoded into a sequence $\langle f_i \mid i < \omega \rangle$ of functions $f_i : (\kappa)^{2i} \to 2$ to which we will apply the principle CF(k). Subsequently a homogeneous sequence for H will mean a homogeneous sequence for $\langle f_i \mid i < \omega \rangle$.

We obtain a natural homogeneous sequence $\langle W_n^* \mid n < \omega \rangle$, $W_n^* = \{x_n, y_n, z_n\}$

for H by successive minimal choices of the x_n: Let $\langle W_i^* \mid i<n\rangle$ be chosen. Let $x_n<\kappa$ be minimal such that there exists a homogeneous sequence $\langle W_i^* \mid i<n\rangle \cup \langle W_i' \mid n\leqslant i<\omega\rangle$ for H with $x_n=\min(W_n')$. Then let $W_n^*=\{x_n, y_n, z_n\}$ be such a W_n'. $\langle W_n^* \mid n<\omega\rangle$ is clearly homogeneous for H.

Set $W_n := \{x_n, y_n\}$, $(n<\omega)$. It is readily seen that $\bigcup\{W_n \mid n<\omega\}$ is a free subset of H.

(1) $[x_n, x_n^+) \cap H[x_n \cup W_{n+1} \cup W_{n+2} \cup \cdots] = \emptyset$.

Proof. Assume not; let $\gamma \in H[x_n \cup W_{n+1} \cup W_{n+2} \cup \cdots]$, $x_n \leqslant \gamma < x_n^+$. There is $f \in H[x_n \cup W_{n+1} \cup W_{n+2} \cup \cdots]$, $f: \omega_r \leftrightarrow \gamma+1$, where $\omega_r = \mathrm{card}(x_n) < x_n$, and hence $x_n \in f''\omega_r \subset H[x_n \cup W_{n+1} \cup W_{n+2} \cup \cdots]$. Let $x_n = t(u_0, \ldots, u_{k-1}, x_{n+1}, y_{n+1}, \ldots, x_l, y_l)$ for some H-term t, $u_0, \ldots, u_{k-1} < x_n$, $l \geqslant n+1$. We may assume that $u_0, \ldots, u_{k-1}$ are successively chosen minimal for this equality to hold. Hence there are H-terms $t_0, \ldots, t_{k-1}$ such that

$$u_j = t_j(x_n, x_{n+1}, y_{n+1}, \ldots, x_l, y_l).$$

Set $u_j' := t_j(y_n, x_{n+1}, y_{n+1}, \ldots, x_l, y_l)$, for $j=0, \ldots, k-1$. Then

$$y_n = t(u_0', \ldots, u_{k-1}', x_{n+1}, y_{n+1}, \ldots, x_l, y_l),$$

and there must exist some $j<k$ such that $u_j \neq u_j'$.

Set $x_n' := u_j$, $y_n' := u_j'$, and $z_n' := t_j(z_n, x_{n+1}, y_{n+1}, \ldots, x_l, y_l)$. Then standard indiscernibility arguments show that x_n', y_n', z_n' are pairwise distinct and that $\langle W_0^*, \ldots, W_{n-1}^*, \{x_n', y_n', z_n'\}, W_{l+1}^*, W_{l+2}^*, \ldots\rangle$ is a homogeneous sequence for H with $\min(\{x_n', y_n', z_n'\}) \leqslant x_n' < x_n = \min(W_n)$. Contradiction. □(1)

(2) Let $\omega_r \leqslant x_n < \omega_{r+1}$. Then $\omega_r < x_n$ and $\sup \mathrm{dom}(U \restriction \omega_{r+1}) < x_n < y_n < \omega_{r+1}$.

Proof. We can assume that ω_r and $\sup \mathrm{dom}(U \restriction \omega_{r+1})$ are constants of the structure H. Then use (1). □(2)

Inside $K[U]$, let $M^* = J_\gamma[F, U] \in K[U]$ be some U-mouse such that $\mathcal{P}(\kappa) \in \mathrm{lp}(M^*)$ and F is countably complete. We can assume that M^* is a constant of H. By 2.6, $M := J_\gamma[U \cup F, \emptyset]$ is a $\emptyset$-mouse inside $K[U]$, and by 2.7, M is a $\emptyset$-mouse in the universe.

For $n<\omega$ set $X_n := H[\bigcup\{W_i \mid n \leqslant i < \omega\}]$. For $m<n<\omega: X_n < X_m$ and $X_n \neq X_m$. Let $\pi_n: H_n \cong X_n$, H_n transitive. For $m \leqslant n < \omega$ let $\pi_{nm} := \pi_m^{-1} \circ \pi_n : H_n \to H_m$. For $n<\omega$ let $\kappa_n := \pi_n^{-1}(\kappa)$, $U_n := \pi_n^{-1}(U)$, $M_n := J_{\gamma_n}[U_n \cup F_n, \emptyset] := \pi_n^{-1}(M)$, $K_n := (K[U_n])^{H_n}$.

(3) $\kappa_n \leqslant \kappa_m$, for $m \leqslant n < \omega$.

M_n is a $\emptyset$-mouse (2.8). $\pi_{nm} \restriction M_n : M_n \to M_m$ is an elementary embedding, and so by 2.12(iii):

(4) $M_n \leqslant_\emptyset^* M_m$, for $m \leqslant n < \omega$.

Let us write $\leqslant^*$ for $\leqslant_\emptyset^*$ and $\sim$ for $\sim_\emptyset$. Since $\leqslant^*$ is a pre-wellordering (2.12(ii)) there exists some $n_0 < \omega$ so that:

(5) $M_n \sim M_{n_0}$ and $\kappa_n = \kappa_{n_0}$ for all $n \geq n_0$.

Fix $n \geq n_0$ so that $\omega_2 \leq \omega_r < x_n < y_n < \omega_{r+1}$. Let $\bar{x}_i := \pi_n^{-1}(x_i)$, $\bar{y}_i := \pi_n^{-1}(y_i)$, $\bar{W}_i := \{\bar{x}_i, \bar{y}_i\}$, for $n \leq i < \omega$. Let $\bar{\omega}_i := \pi_n^{-1}(\omega_i)$, for $i < \omega$.

Now set $\tilde{X} := H_n[\bar{x}_n \cup \bigcup\{\bar{W}_i \mid n+1 \leq i < \omega\}]$. Let $\tilde{\pi} : \tilde{H} \cong \tilde{X}$, $\tilde{H}$ transitive. Let $\tilde{\kappa} := \tilde{\pi}^{-1}(\kappa_n)$, $\tilde{U} := \tilde{\pi}^{-1}(U_n)$, $\tilde{K} := (K[\tilde{U}])^{\tilde{H}}$, and $\tilde{M} := J_{\tilde{\gamma}}[\tilde{U} \cup \tilde{F}, \emptyset] := \tilde{\pi}^{-1}(M_n)$. $\tilde{M}$ is a $\emptyset$-mouse (2.8). By (1),

(6) $\bar{x}_n \subset \tilde{X}$ and $[\bar{x}_n, \bar{\omega}_{r+1}) \cap \tilde{X} = \emptyset$.

Hence:

(7) $\bar{x}_n$ is the critical point of $\tilde{\pi}$ and $\tilde{\pi}(\bar{x}_n) = \bar{\omega}_{r+1}$.

There is a unique elementary map $\tilde{\sigma} : H_{n+1} \to \tilde{H}$ such that $\tilde{\pi} \circ \tilde{\sigma} = \pi_{n+1,n}$:

$$\begin{tikzcd} H_{n+1} \arrow[rr, "\pi_{n+1,n}"] \arrow[dr, "\tilde{\sigma}"'] & & H_n \\ & \tilde{H} \arrow[ur, "\tilde{\pi}"'] & \end{tikzcd}$$

$\tilde{\sigma}$ is determined by: $\pi_{n+1}^{-1}(x_i) \mapsto \tilde{\pi}^{-1}(\bar{x}_i)$ and $\pi_{n+1}^{-1}(y_i) \mapsto \tilde{\pi}^{-1}(\bar{y}_i)$, $(n+1 \leq i < \omega)$. $\tilde{\sigma} \restriction M_{n+1} : M_{n+1} \to \tilde{M}$, $\tilde{\pi} \restriction \tilde{M} : \tilde{M} \to M_n$ are elementary embeddings, and so by 2.12(iii) and (5),

(8) $\tilde{M} \sim M_n$.

Since $\tilde{\pi} \restriction \bar{x}_n = \mathrm{id}$,

(9) $(\tilde{U} \cup \tilde{F}) \restriction \bar{x}_n \cap \tilde{M} \cap M_n = (U_n \cup F_n) \restriction \bar{x}_n \cap \tilde{M} \cap M_n$.

By 2.12(iv),

(10) $\mathcal{P}(\bar{x}_n) \cap \tilde{K} = \mathcal{P}(\bar{x}_n) \cap \tilde{M} = \mathcal{P}(\bar{x}_n) \cap M_n = \mathcal{P}(\bar{x}_n) \cap K_n$.

By (10), and because $\bar{x}_n$ is the critical point of $\tilde{\pi} \restriction \tilde{K} : \tilde{K} \to K_n$:

(11) $\bar{x}_n$ is weakly inaccessible in $\tilde{K}$.

(12) $\bar{x}_n$ is weakly inaccessible in K_n, and so x_n and y_n are weakly inaccessible in $K[U]$.

(13) $\mathrm{cof}(x_n) = \mathrm{cof}(y_n) = \omega_r$.

Proof. Assume $\mathrm{cof}(x_n) < \omega_r$. $\omega_2 \leq x_n$ and $\sup \mathrm{dom}(U \restriction \omega_{r+1}) < x_n$ (2). By the covering property 3.20(i), x_n is singular in $K[U]$, contradicting (12). $\square$(13)

(14) $x_n \notin H[\omega_{r-1} \cup \{y_n, x_{n+1}, y_{n+1}, \ldots\}]$.

Proof. Assume $x_n = t(\vec{\eta}, y_n, x_{n+1}, y_{n+1}, \ldots, x_l, y_l)$ for some H-term t, $\vec{\eta} < \omega_{r-1}$, and $l \geq n+1$. Since $\mathrm{cof}(x_n) = \omega_r$,

$$x_n' := \sup(x_n \cap \{t(\vec{\nu}, x_n, x_{n+1}, y_{n+1}, \ldots, x_l, y_l) \mid \vec{\nu} < \omega_{r-1}\}) < x_n.$$

Set

$$y'_n := \sup(y_n \cap \{t(\vec{v}, y_n, x_{n+1}, y_{n+1}, \dots, x_l, y_l) \mid \vec{v} < \omega_{r-1}\}), \quad \text{and}$$
$$z'_n := \sup(z_n \cap \{t(\vec{v}, z_n, x_{n+1}, y_{n+1}, \dots, x_l, y_l) \mid \vec{v} < \omega_{r-1}\}).$$

Since $x_n = t(\vec{\eta}, y_n, x_{n+1}, y_{n+1}, \dots, x_l, y_l)$, we obtain $x'_n < x_n < y'_n$. By simple indiscernibility arguments, x'_n, y'_n, z'_n are pairwise distinct and $\langle W_0^*, \dots, W_{n-1}^*, \{x'_n, y'_n, z'_n\}, W_{l+1}^*, W_{l+2}^*, \dots \rangle$ is a homogeneous sequence for H. This contradicts the minimal choice of x_n. (14)

Set $\hat{X} := H_n[\bar{\omega}_{r-1} \cup \{\bar{y}_n, \bar{x}_{n+1}, \bar{y}_{n+1}, \dots\}]$. We shall carry out an analysis of $\hat{X}$ similar to the preceding one of $\bar{X}$. Let $\hat{\pi} : \hat{H} \cong \hat{X}$, $\hat{H}$ transitive. Let $\hat{\kappa} := \hat{\pi}^{-1}(\kappa_n)$, $\hat{U} := \hat{\pi}^{-1}(U_n)$, $\hat{K} := (K[\hat{U}])^{\hat{H}}$, and $\hat{M} := J_{\hat{\gamma}}[\hat{U} \cup \hat{F}, \emptyset] := \hat{\pi}^{-1}(M_n)$. $\hat{M}$ is a $\emptyset$-mouse (2.8). By (14),

(15) $\bar{x}_n \notin \hat{X}$.

(16) $\bar{\omega}_r \not\subset \hat{X}$.

Proof. Assume $\bar{\omega}_r \subset \hat{X}$. Take $f \in \hat{X}$, $f : \bar{\omega}_r \leftrightarrow \bar{y}_n$. Then $\bar{x}_n \in \bar{y}_n = f''\bar{\omega}_r \subset \hat{X}$, contradicting (15). □(16)

Let α be the critical point of $\hat{\pi}$.

(17) $\bar{\omega}_{r-1} < \alpha < \bar{\omega}_r$, $[\alpha, \bar{\omega}_r) \cap \hat{X} = \emptyset$, $\hat{\pi}(\alpha) = \bar{\omega}_r$.

Proof. Assume $\beta \in \hat{X}$, $\alpha \leq \beta < \bar{\omega}_r$. Take $f \in \hat{X}$, $f : \bar{\omega}_{r-1} \leftrightarrow \beta + 1$. Then $\alpha \in \beta + 1 = f''\bar{\omega}_{r-1} \subset \hat{X}$. Contradiction. □(17)

In particular,

(18) $\sup \operatorname{dom}(U_n \restriction \bar{\omega}_r) < \alpha$.

As in the discussion of $\bar{\pi} : \bar{H} \to H_n$ there is a unique elementary map $\hat{\sigma} : H_{n+1} \to \hat{H}$ such that $\hat{\pi} \circ \hat{\sigma} = \pi_{n+1,n}$:

$$\begin{tikzcd} H_{n+1} \arrow[rr, "\pi_{n+1,n}"] \arrow[dr, "\hat{\sigma}"'] & & H_n \\ & \hat{H} \arrow[ur, "\hat{\pi}"'] & \end{tikzcd}$$

And as before we obtain:

(19) $\hat{M} \sim M_n$.

(20) $(\hat{U} \cup \hat{F}) \restriction \alpha \cap \hat{M} \cap M_n = (U_n \cap F_n) \restriction \alpha \cap \hat{M} \cap M_n$.

(21) $\mathcal{P}(\alpha) \cap \hat{K} = \mathcal{P}(\alpha) \cap K_n$.

(22) α is weakly inaccessible in $\hat{K}$.

(23) $\bar{\omega}_r$ is weakly inaccessible in K_n, and ω_r is weakly inaccessible in $K[U]$.

By (12), $\bar{\omega}_r^{+K_n} < \bar{x}_n < \bar{\omega}_{r+1} = \bar{\omega}_r^{+H_n}$, and since $\hat{\pi}(\alpha) = \bar{\omega}_r$:

(24) $\alpha^{+\hat{K}} < \alpha^{+\hat{H}}$.

(25) $\mathcal{P}(\bar{\omega}_r) \cap K_n \subset \hat{K}$.

Proof. Inside $\hat{K}$ form an iterate M^+ of $\hat{M}$ at points $\geq \alpha$ so that $\min(\text{meas}(M^+) \setminus \alpha) > \hat{\kappa} = \kappa_n > \bar{\omega}_r$. Since α is the critical point of $\hat{\pi}$ the ultrafilters of M^+ and M_n agree for all $\nu < \bar{\omega}_r$. $M^+ \sim \hat{M} \sim M_n$, and by 2.12(iv),

$$\mathcal{P}(\bar{\omega}_r) \cap K_n = \mathcal{P}(\bar{\omega}_r) \cap M_n = \mathcal{P}(\bar{\omega}_r) \cap M^+ \subset \hat{K}. \qquad \square(25)$$

Set $D := \{u \subset \alpha \mid u \in \hat{K} \wedge \alpha \in \hat{\pi}(u)\}$. D is a normal ultrafilter on $\mathcal{P}(\alpha) \cap \hat{K}$.

(26) $\text{Ult}(\hat{K}, D)$ is well-founded.

Proof. $[f]_D \mapsto \hat{\pi}(f)(\alpha)$ defines a Σ_0-preserving embedding of $\text{Ult}(\hat{K}, D)$ into K_n. $\square(26)$

(27) $D \in H_n$.

Proof. By (24), $\alpha^{+\hat{K}} < \alpha^{+\hat{H}}$. Since $\hat{K}$ satisfies the GCH (3.7), there is $g \in \hat{H}$, $g : \alpha \to \mathcal{P}(\alpha) \cap \hat{K}$ onto. Then

$$\begin{aligned} D &:= \{g(\nu) \mid \nu < \alpha \wedge \alpha \in \hat{\pi}(g(\nu))\} \\ &= \{(\hat{\pi}(g)(\nu)) \cap \alpha \mid \nu < \alpha \wedge \alpha \in \hat{\pi}(g)(\nu)\} \in H_n. \qquad \square(27) \end{aligned}$$

(28) Set $\bar{K} := (K_{\bar{\omega}_r}[U_n])^{H_n}$. Then $\text{Ult}(\bar{K}, D)$ is well-founded.

Proof. By (25), $\bar{K} \subset \hat{K}$, and $\text{Ult}(\hat{K}, D)$ is well-founded by (26). $\square(28)$

This fact holds as well inside H_n. Let $\alpha^* := \pi_n(\alpha)$, $D^* := \pi_n(D)$. Since $\pi_n : H_n \to H$ is elementary, $\text{Ult}(K_{\omega_r}[U], D^*)$ is well-founded. By 3.25, there is an elementary embedding $j : K[U \restriction \omega_r] \to K[U \restriction \omega_r]$ with critical point α^*. $\alpha^* > \sup \text{dom}(U \restriction \omega_r)$ (by (18)), ω_r is regular and weakly inaccessible in $K[U \restriction \omega_r]$ (23). By the embedding Theorem 3.18 there is a strong $U^* >_e U \restriction \omega_r$ with $\min \text{dom}(U^* \setminus \alpha^*) < \omega_r$. But this is a *contradiction* to U being the canonical sequence. $\square$

8. Non-closure of the image model

8.1. Theorem. *Assume there is an elementary embedding $\pi : V \to M$, M transitive with critical point κ such that ${}^{\omega}M \subset M$ and ${}^{\kappa}M \not\subset M$. Then 0^{long} exists.*

This strengthens a theorem of Sureson's [20] who from the same assumption could show the existence of an inner model with ω_1 measurable cardinals.

Proof. Assume $\neg 0^{\text{long}}$ and work for a contradiction. Let $U_0 := U_{\text{can}}$, $U' := \pi(U_{\text{can}})$, and $\bar{\pi} := \pi \restriction K[U_0]$. $\bar{\pi} : K[U_0] \to K[U']$ is elementary, and by 3.17, $\bar{\pi}$ is the iteration map of a normal iterated ultrapower of $K[U_0]$. This means that there is a strictly increasing index $\langle \kappa_i \mid i<\delta\rangle$ such that if $\langle\langle K[U_i] \mid i\leq\delta\rangle, \langle \pi_{ij} \mid i\leq j\leq \delta\rangle\rangle$ is the corresponding iteration of $K[U_0]$ by $\langle \kappa_i \mid i<\delta\rangle$, then $K[U']=K[U_\delta]$ and $\bar{\pi}=\pi_{0\delta}$. We may of course assume that for every $i<\delta$, κ_i is a measurable cardinal in $K[U_i]$, $\kappa_i\in \mathrm{dom}(U_i)$. Then $\bar{\kappa}_0=\kappa$, and since $\neg 0^{\mathrm{long}}$,

(1) $\mathrm{otp}\, \mathrm{dom}(U_0)<\kappa$.

The following claim is taken over from Sureson [20] with a slightly different presentation of the proof:

(2) $\delta\geq\omega$.

Proof. Assume $\delta<\omega$. Let $X:=\{\pi(f)(\kappa_0,\ldots,\kappa_{\delta-1}) \mid f:\kappa_0\times\cdots\times\kappa_{\delta-1}\to V\}$.

(2′) $X=M$.

Proof. For every $\gamma\in\mathrm{On}$, $\gamma=\pi_{0\delta}(f)(\kappa_0,\ldots,\kappa_{\delta-1})=\pi(f)(\kappa_0,\ldots,\kappa_{\delta-1})$ for some $f\in K[U_0]$, $f:\kappa_0\times\cdots\times\kappa_{\delta-1}\to\mathrm{On}$ by the 'representation properties' of the iterated ultrapower $K[U_\delta]$ (compare Definition 2.2(iii)). Hence $\mathrm{On}\subset X$.

Let $z\in M$. Let $z\in V_\alpha$ and take $g:\beta\to V_\alpha$ onto. Then $\pi(g):\pi(\beta)\to V^M_{\pi(\alpha)}$ onto; let $z=\pi(g)(\gamma)$ for some $\gamma\in\mathrm{On}$. $\gamma=\pi(f)(\kappa_0,\ldots,\kappa_{\delta-1})$ for some $f:\kappa_0\times\cdots\times\kappa_{\delta-1}\to V$. Define $h:\kappa_0\times\cdots\times\kappa_{\delta-1}\to V$ by: $h(x_0,\ldots,x_{\delta-1})=g(f(x_0,\ldots,x_{\delta-1}))$ if this is defined, and $h(x_0,\ldots,x_{\delta-1})=0$ else. Then

$$z=\pi(g)(\pi(f)(\kappa_0,\ldots,\kappa_{\delta-1}))=\pi(h)(\kappa_0,\ldots,\kappa_{\delta-1})\in X. \qquad \square(2')$$

(2″) ${}^{\kappa}M\subset M$.

Proof. Let $\{x_\alpha \mid \alpha<\kappa\}\subset M$. Choose a sequence $\langle f_\alpha \mid \alpha<\kappa\rangle$, $f_\alpha:\kappa_0\times\cdots\times\kappa_{\delta-1}\to V$ such that $x_\alpha=\pi(f_\alpha)(\kappa_0,\ldots,\kappa_{\delta-1})$ for $\alpha<\kappa$. Define $F:\kappa_0\times\cdots\times\kappa_{\delta-1}\to V$ by $F(x_0,\ldots,x_{\delta-1})=\langle f_\alpha(x_0,\ldots,x_{\delta-1}) \mid \alpha<\kappa\rangle$. For $\alpha<\kappa$, $(\pi(F)(\kappa_0,\ldots,\kappa_{\delta-1}))(\alpha)=\pi(f_\alpha)(\kappa_0,\ldots,\kappa_{\delta-1})=x_\alpha$. So

$$\langle x_\alpha \mid \alpha<\kappa\rangle=(\pi(F)(\kappa_0,\ldots,\kappa_{\delta-1}))\restriction\kappa\in M. \qquad \square(2'')$$

But (2″) contradicts the non-closure property of M. $\square(2)$

Let C be a Prikry system for $K[U_{\mathrm{can}}]$ which satisfies the covering Theorem 3.23. Let $C_0:=\bigcup\{C(\gamma) \mid \gamma\in\mathrm{dom}(U_{\mathrm{can}})\}$. Set $C':=\pi(C)$, $C_0':=\pi(C_0)$. By (1), the ordertype of C_0' is $<\kappa$, and we get

(3) $C_0'=\pi''C_0\subset\mathrm{range}(\pi)$.

Set $\tau:=\sup\{\kappa_i \mid i<\omega\}$, $X:=\{\kappa_i \mid i<\omega\}$, and $D:=C_0'\cap\tau$. $X\in M$ since ${}^{\omega}M\subset M$. In M, apply the covering Theorem 3.23 to X: There is $f:(\tau)^{<\omega}\to\tau$, $f\in K[U']$ and a $\gamma<\tau$ such that

(4) $X\subset\{f(\vec{\nu},\vec{\mu}) \mid \vec{\nu}<\gamma, \vec{\mu}\in D\}$.

Since the iteration by $\langle \kappa_i \mid i<\delta\rangle$ is normal, $\mathcal{P}(\tau)\cap K[U_\omega]=\mathcal{P}(\tau)\cap K[U_\delta]$ (compare 2.4(iii)). So $f\in K[U_\omega]$. Choose $i<\omega$ such that $\kappa_i>\gamma$ and $f\in$

range($\pi_{i\omega}$); let $f = \pi_{i\omega}(\bar{f})$. By (4), $\kappa_i = f(\vec{v}, \vec{\mu})$ for some $\vec{v} < \kappa_i$, $\vec{\mu} \in D$. By (3), $D \subset$ range($\pi_{i\omega}$); let $\vec{\mu} = \pi_{i\omega}(\bar{\vec{\mu}})$. Then

$$\kappa_i = \pi_{i\omega}(\bar{f})(\vec{v}, \pi_{i\omega}(\bar{\vec{\mu}})) = \pi_{i\omega}(\bar{f}(\vec{v}, \bar{\vec{\mu}})) \in \mathrm{range}(\pi_{i\omega}).$$

But since κ_i is the critical point of $\pi_{i\omega}$ this is impossible. □

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