Homework problems (due June 5)

Problem 1 (A non-reduced curve)

Consider the 1-dimensional k-scheme $X = V_+(Z^2) \subseteq \mathbb{P}^2_k$. It is non-reduced; let $\mathcal{N} \subset \mathcal{O}_X$ be the ideal sheaf of nilpotent elements and let $X_{\mathrm{red}} = V(\mathcal{N}) = V_+(Z)$ be the maximal reduced closed subscheme.

(a) Prove that $\mathcal{N}$ is naturally a quasi-coherent $\mathcal{O}_{X_{\mathrm{red}}}$-module. Show that it is locally free of rank 1 as $\mathcal{O}_{X_{\mathrm{red}}}$-module.

(b) Determine $H^0(X, \mathcal{O}_X)$, $H^0(X, \mathcal{N})$ and the degree $\deg(\mathcal{N})$ of $\mathcal{N}$ as line bundle on X_{red}.

Hint: You may freely use results about the cohomology of $\mathcal{O}_{\mathbb{P}^2_k}(-2)$ as in §30.8, Tag 01XT, of the Stacks project.

Problem 2 (Relative effective Cartier divisors)

Let $X \to S$ be a morphism of schemes and let $Z \subseteq X$ be the closed subscheme defined by an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$.

(a) Consider a base change diagram

$$\begin{tikzcd} X_T \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ T \arrow[r] & S. \end{tikzcd}$$

Let $Z_T \subseteq X_T$ be the base change of Z and denote by $\mathcal{I}_T$ its ideal sheaf. Prove that if Z is flat over S, then the natural map defines an isomorphism

$$f^*\mathcal{I} \xrightarrow{\sim} \mathcal{I}_T.$$

(b) We call Z an effective Cartier divisor of X if it is locally defined by the vanishing of a single equation that is not a zero-divisor. Show that this condition is equivalent to $\mathcal{I}$ being a line bundle on X.

(c) Show that if Z is an effective Cartier divisor and flat over S, then $Z_T \subseteq X_T$ is an effective Cartier divisor as well.

Further Problems

Problem 3 (The Legendre family)

Let $S = \operatorname{Spec} k[\lambda]$ and $X = V(y^2 - x(x-1)(x-\lambda)) \subseteq \mathbb{A}^2_S$. Determine the maximal open subscheme $S_{\mathrm{sm}} \subset S$ that has the property that $S_{\mathrm{sm}} \times_S X \to S$ is smooth.