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In this note we review Austin's proof [Aus11] of the density Hales-Jewett (DHJ) theorem [FK91] via sated extensions. After translating DHJ into a measure-theoretic problem Furstenberg and Katznelson obtained a suitable invariance of the measure using the Carlson-Simpson theorem. In this note we shall use the Graham-Rothschild theorem instead.

In order to make the notation less heavy we identify natural numbers with finite von Neumann ordinals, i.e. $k = \{0, \dots, k-1\}$. Expressions like k^n usually denote not the integer exponentiation but the set-theoretic one, thus k^n is the set of functions from $\{0, \dots, n-1\}$ to $\{0, \dots, k-1\}$ or the set of words of length n on the alphabet $\{0, \dots, k-1\}$. The integer expression k^n is the cardinality of our set k^n.

1 Density Hales-Jewett

Let (X, Σ) be a standard Borel space. A *law* is a sequence of Borel probability measures $(\mu^n)_{n=0}^{\infty}$ on $(X^{k^n}, \Sigma^{\otimes k^n})$. Note that Borel probability measures on Polish spaces are Radon (TODO: find some measure theory book where it is proved), thus form a weakly compact metrizable set.

A law $\tilde{\mu}$ on $(\tilde{X}, \tilde{\Sigma})$ together with a Borel map $\varphi : \tilde{X} \to X$ is an *extension* of μ if for every n we have $(\varphi^{\otimes k^n})_* \tilde{\mu} = \mu$.

1.1 Stationary laws

For an n-dimensional combinatorial subspace $S^n : k^n \to k^m$ one has a pullback map $S^n_* : X^{k^m} \to X^{k^n}$ and a pushforward map $S^n_{**} : M(X^{k^m}) \to M(X^{k^n})$. A law is called *strongly stationary (s.s.)* if for every n and every n-dimensional combinatorial subspace $S^n : k^n \to k^m$ one has $S^n_{**}\mu^m = \mu^n$.

Lemma 1. *For every law μ there exists a sequence of combinatorial subspaces (S_m) such that the laws $(S_m)_{**}\mu$ converge in the product weak topology (i.e. each of the measures constituting these laws converges weakly) to a s.s. law as $m \to \infty$.*

Proof. Pick a sequence of natural numbers $(n_m)_m$ in which each number occurs infinitely often and pick a metric on the compact metrizable space $M(X^{k^n})$ for each n. We construct the sequence $(S_m)_m$ inductively as follows. Begin with $S_0 = \mathrm{id}$.

At step $m > 0$ partition the compact metric space $M(X^{k^{n_m}})$ into finitely many sets of diameter at most 2^{-m}. To each n_m-dimensional subspace $S^{n_m} : k^{n_m} \to k^N$ associate the measure $(S^{n_m})_{**}((S_{m-1})_{**}\mu)^N$. By the Graham-Rothschild theorem there exists a subspace $S = (S^n : k^n \to k^{N(n)})_n$ such that, for every n, the measures associated to all n_m-dimensional subspaces of each $S^n(k^n)$ lie in the same cell of the partition.

By the pigeonhole principle one of the cells occurs infinitely often. Replacing each S^n by a subspace of some $S^{n'}$ with $n' \geq n$ that corresponds to this cell we may assume that the cell does not depend on n. This property is evidently preserved under taking subspaces.

Define $S_m = S_{m-1} \circ S$.

Since each n occurs infinitely often in the sequence $(n_m)_m$, we obtain for the resulting sequence of subspaces that the measures corresponding to n-dimensional subspaces eventually lie in a 2^{-m}-ball in $M(X^{k^n})$, for any $m \in \mathbb{N}$.

This shows that $((S_m)_{**}\mu)^n$ does converge weakly for each n and that the limit is s.s. (since the pushforward operation on measures is weakly continuous). $\square$

1.2 Relative independence

Proposition 2 (See [Tao07, Appendix]). *Two σ-algebras B_1 and B_2 are called relatively independent over B (under a measure μ) if one of the following equivalent conditions holds:*

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1. *For all* $f_1 \in L^\infty(B_1)$*,* $f_2 \in L^\infty(B_2)$ *we have* $\mathbb{E}(f_1 f_2|B) = \mathbb{E}(f_1|B)\mathbb{E}(f_2|B)$*,*
2. *For all* $f_1 \in L^1(B_1)$ *we have* $\mathbb{E}(f_1|B \vee B_2) = \mathbb{E}(f_1|B)$*,*
3. *For all* $f_1 \in L^2(B_1)$ *we have* $\|\mathbb{E}(f_1|B \vee B_2)\|_2 = \|\mathbb{E}(f_1|B)\|_2$*,*
4. *For all characteristic functions* $f_1 \in L^2(B_1)$ *we have* $\|\mathbb{E}(f_1|B \vee B_2)\|_2 = \|\mathbb{E}(f_1|B)\|_2$*.*

Proposition 3. *If* $B \subset B_1$ *then the above statements are equivalent to any of the following.*

1. *For all* $f_1 \in L^2(B_1)$ *such that* $f_1 \perp B$ *we have* $f_1 \perp B \vee B_2$*,*
2. *For all* $f_1 \in L^2(B_1)$ *such that* $f_1 \not\perp B \vee B_2$ *we have* $f_1 \not\perp B$*.*

Proof. The third statement above clearly implies the first here, and the two statements are clearly equivalent. Assuming the first statement take $f_1 \in L^2(B_1)$ and write

$$f_1 = \mathbb{E}(f_1|B) + (f_1 - \mathbb{E}(f_1|B)).$$

By definition of the conditional expectation the expression in parentheses is orthogonal to B and by the assumption $B \subset B_1$ it is B_1-measurable. Hence it is also orthogonal to $B \vee B_2$. Since $\mathbb{E}(f_1|B)$ is also $B \vee B_2$-measurable, we obtain $\mathbb{E}(f_1|B \vee B_2) = \mathbb{E}(f_1|B)$. By density this implies the second statement above. □

1.3 Sated laws

Definition 4. Let μ be a s.s. law and $e \subset k$. The *e-insensitive* σ-algebra $\Sigma_e \subset \Sigma$ is defined by

$$A \in \Sigma_e :\iff \mu^1(\pi_i^{-1}(A)\Delta\pi_j^{-1}(A)) = 0 \quad \text{for every } i, j \in e.$$

An *up-set* is a subset $I \subset \binom{k}{\geq 1}$ such that if $e \in I$ and $e \subset e' \subset k$ then $e' \in I$. For an up-set I define $\Sigma_I := \vee_{e \in I} \Sigma_e$.

A s.s. law is called *I-sated* if for every s.s. extension $\varphi : \tilde{\mu} \to \mu$ the algebras $\varphi^{-1}(\Sigma)$ and $\tilde{\Sigma}_I$ are relatively independent over $\varphi^{-1}(\Sigma_I)$. A s.s. law is called *fully sated* if it is sated for every I.

Proposition 5. *Every s.s. law* μ *admits a fully sated extension.*

Proof. The fully sated extension is constructed as an inverse limit of extensions $(X_{(m)}, \Sigma_{(m)}, \mu_{(m)})$ starting with $(X_{(0)}, \Sigma_{(0)}, \mu_{(0)}) = (X, \Sigma, \mu)$. Let also $(f_{(0,p)})_p$ be a dense sequence in $L^2(X, \Sigma, \mu^0)$. Let $(r_m, p_m, I_m)_m$ be a sequence in $\mathbb{N} \times \mathbb{N} \times \binom{k}{\geq 1}$ such that $r_m \leq m$ and such that each triple occurs infinitely often.

Let $(X_{(m)}, \Sigma_{(m)}, \mu_{(m)})$ be given and choose a s.s. extension

$$\varphi_{(m+1)} \colon (X_{(m+1)}, \Sigma_{(m+1)}, \mu_{(m+1)}) \to (X_{(m)}, \Sigma_{(m)}, \mu_{(m)})$$

in such a way that $\|\mathbb{E}(f_{(r_m,p_m)}|\Sigma_{(m+1),I_m})\|_2$ differs from its maximal possible value by at most 2^{-m}. Let also $(f_{(m+1,p)})_p$ be a dense sequence in $L^2(X_{(m+1)}, \Sigma_{(m+1)}, \mu^0_{(m+1)})$.

Let $(X_{(\omega)}, \Sigma_{(\omega)}, \mu_{(\omega)})$ be the inverse limit of this tower of extensions. (TODO: verify strong stationarity) We claim that this inverse limit is fully sated.

Indeed, let $\tilde{\varphi} : (\tilde{X}, \tilde{\Sigma}, \tilde{\mu}) \to (X_{(\omega)}, \Sigma_{(\omega)}, \mu_{(\omega)})$ be a s.s. extension. We have to verify that, for every $f \in L^\infty(X_{(\omega)}, \Sigma_{(\omega)}, \mu^0_{(\omega)})$ and $I \subset \binom{k}{\geq 1}$, we have

$$\mathbb{E}(f \circ \tilde{\varphi}|\tilde{\Sigma}_I) = \mathbb{E}(f|\Sigma_{(\omega),I}) \circ \tilde{\varphi}.$$

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By the martingale convergence theorem and continuity of the expectation operator it suffices to consider functions $f = f_{(r,q)}$.

Let m be such that $r = r_m$, $q = q_m$, $I = I_m$. By definition of $X_{(m+1)}$ we have

$$\|\mathbb{E}(f \circ \tilde{\varphi}|\tilde{\Sigma}_I)\|_2 \leq 2^{-m} + \|\mathbb{E}(f \circ \varphi_{(m+1)}|\Sigma_{(m+1),I})\|_2 \leq 2^{-m} + \|\mathbb{E}(f \circ \varphi_{(\omega)}|\Sigma_{(\omega),I})\|_2.$$

Since m can be chosen arbitrarily large, this implies

$$\|\mathbb{E}(f \circ \tilde{\varphi}|\tilde{\Sigma}_I)\|_2 \leq \|\mathbb{E}(f \circ \varphi_{(\omega)}|\Sigma_{(\omega),I})\|_2.$$

Since $\tilde{\Sigma}_I \supset \tilde{\varphi}^{-1}(\Sigma_{(\omega),I})$, this implies

$$\mathbb{E}(f \circ \tilde{\varphi}|\tilde{\Sigma}_I) = \mathbb{E}(f \circ \varphi_{(\omega)}|\Sigma_{(\omega),I}) \circ \tilde{\varphi}$$

as required. □

1.4 Joining topology

Let $(X_i, \mu_i)_i$ be a countable family of compact Hausdorff spaces with probability Borel measures. The Borel probability measures on $\prod_i X_i$ whose i-th marginals equal μ_i are called joining measures. The joining topology on the space of joining measures is the coarsest topology for which each map

$$\mu \mapsto \int \prod_i f_i \circ \pi_i \mathrm{d}\mu, \quad f_i \in L^\infty(X_i, \mu_i)$$

is continuous. This topology is compact since it suffices to consider functions from a countable L^1-dense set of the unit ball of $L^\infty(X_i, \mu_i)$ for each i.

Using compactness of the joining topology we obtain the following analog of Lemma 1.

Lemma 6. *Let λ be a joining of countably many s.s. laws $(\mu_j)_j$ and S_0 be a combinatorial subspace.*

Then there exists a sequence of nested combinatorial subspaces $(S_m)_m$ starting with S_0 such that the laws $(S_m)_{**}\lambda$ converge in the joining topology to a s.s. joining of $(\mu_j)_j$ as $m \to \infty$.

1.5 Relatively independent σ-algebras associated to fully sated laws

We begin with the construction that will be used to exploit the satedness of a given law.

Definition 7. Let $e \subset k$. The *oblique copy* $\Sigma_e^\dagger$ of Σ_e is (the μ^1-completion of) the σ-algebra $\pi_i^{-1}(\Sigma_e)$ for any $i \in e$ (this σ-algebra clearly does not depend on $i \in e$).

For an up-set I let $\Sigma_I^\dagger := \vee_{e \in I} \Sigma_e^\dagger$.

The key feature of fully sated laws is the following independence property they enjoy.

Proposition 8. *Let μ be fully sated s.s. law and $e, e' \subset k$ be disjoint. Then the oblique σ-algebras $\Sigma_e^\dagger$ and $\vee_{j \in e'} \Sigma_{\{j\}}^\dagger$ are relatively independent over $\vee_{j \in e'} \Sigma_{e \cup \{j\}}^\dagger$ under μ^1.*

Proof. It suffices to show that whenever $f \circ \pi_i \in L^2(\Sigma_e^\dagger)$ (where $i \in e$) is not orthogonal to $\vee_{j \in e'} \Sigma_{\{j\}}^\dagger$ it is also not orthogonal to $\vee_{j \in e'} \Sigma_{e \cup \{j\}}^\dagger$. By the assumption we have

$$0 \neq \kappa = \int_{X^k} f \circ \pi_i \prod_{j \in e'} h_j \circ \pi_j \mathrm{d}\mu^1$$

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for some measurable functions h_j on X and every $i \in e$. By strong stationarity we have

$$\kappa = \int_{X^{k^n}} f(x_{r_{e,i}(w)}) \prod_{j\in e'} h_j(x_{r_{e,j}(w)}) \mathrm{d}\mu^n(x)$$

for any n and every word $w \in k^n$ in which a letter from e occurs. Here $r_{e,i}$ is the operation on words that replaces each letter from e by the letter i. Using the assumptions on f we obtain

$$\kappa = \int_{X^{k^n}} f(x_w) \prod_{j\in e'} h_j(x_{r_{e,j}(w)}) \mathrm{d}\mu^n(x)$$

for all w in which a letter from e occurs. Let S_0 be a subspace in which every word begins with $i \in e$ (say), so that the above equality holds for every word in this subspace.

Now we construct an extension of X as follows. Let $\tilde{X} := X \times X^{e'}$ with coordinate projections θ_e, $(\theta_j)_{j\in e'}$. Let also

$$\iota^n : X^{k^n} \to \tilde{X}^{k^n}, \quad (x_w)_w \mapsto (x_w, (x_{r_{e,j}(w)})_{j\in e'})_w$$

and $\lambda^n = (\iota^n)_*\mu^n$ be the pushforward measures.[1]

Write $\tilde{x} = (\tilde{x}_w)_{w\in k^n} = (x_w, (z_{w,j})_{j\in e'})_{w\in k^n}$ for elements of $\tilde{X}^{k^n}$.

Let $j \in e'$ and $w \in k^n$ be arbitrary. By definition of λ^n we have

$$(\theta_j \circ \pi_w)\tilde{x} = z_{w,j} = x_{r_{e,j}(w)} = (\pi_{r_{e,j}(w)} \circ \theta_e)\tilde{x} \tag{9}$$

for λ^n-a.e. $\tilde{x}$, so that

$$(\theta_j \circ \pi_w)_*\lambda^n = (\pi_{r_{e,j}(w)} \circ \theta_e)_*\lambda^n = (\pi_{r_{e,j}(w)})_*\mu^n = \mu^0$$

by strong stationarity of μ. We also trivially have $\theta_e \circ \pi_w = \pi_w \circ \theta_e$, so that

$$(\theta_e \circ \pi_w)_*\lambda^n = (\pi_w \circ \theta_e)_*\lambda^n = (\pi_w)_*\mu^n = \mu^0.$$

Thus every marginal of λ^n on a copy of X equals μ^0.

By Lemma 6 there exists a sequence of nested combinatorial subspaces $(S_m)_m$ contained in S_0 such that $(S_m)_{**}\lambda$ converges in the coupling topology to a s.s. law $\tilde{\mu}$ as $m \to \infty$. Then

$$\theta_e : (\tilde{X}, \tilde{\Sigma}, (S_m)_{**}\lambda) \to (X, \Sigma, (S_m)_{**}\mu)$$

is an extension for every m, and since $(S_m)_{**}\mu = \mu$ by strong stationarity, the limit law is again an extension

$$\theta_e : (\tilde{X}, \tilde{\Sigma}, \tilde{\mu}) \to (X, \Sigma, \mu).$$

Recalling (9) we obtain

$$\int_{\tilde{X}^{k^n}} f\circ\theta_e\circ\pi_w \prod_{j\in e'} h_j\circ\theta_j\circ\pi_w \mathrm{d}\lambda^n = \int_{\tilde{X}^{k^n}} f\circ\pi_w\circ\theta_e \prod_{j\in e'} h_j\circ\pi_{r_{e,j}(w)}\circ\theta_e \mathrm{d}\lambda^n = \int_{X^{k^n}} f\circ\pi_w \prod_{j\in e'} h_j\circ\pi_{r_{e,j}(w)} \mathrm{d}\mu^n = \kappa$$

for every word w in the subspace S_0. Since each S_m is a subspace of S_0 and by convergence in the coupling topology we have

$$\int_{\tilde{X}} f\circ\theta_e \underbrace{\prod_{j\in e'} h_j \circ \theta_j}_{=:\tilde{h}} \mathrm{d}\tilde{\mu}^0 = \lim_m \int_{\tilde{X}} f\circ\theta_e \prod_{j\in e'} h_j\circ\theta_j \mathrm{d}((S_m^0)_{**}\lambda^n) = \lim_m \int_{\tilde{X}^{k^n}} f\circ\theta_e\circ\pi_{S_m^0()} \prod_{j\in e'} h_j\circ\theta_j\circ\pi_{S_m^0()} \mathrm{d}\lambda^n = \kappa.$$

[1] $\theta_e^{\otimes k^n}$ is the inverse of ι^n up to sets of measure zero

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Let $i \in e$, $j \in e'$ be arbitrary. By (9) we have

$$\theta_j \circ \pi_{S^1_m(i)} = \pi_{r_{e,j}(S^1_m(i))} \circ \theta_e = \pi_{r_{e,j}(S^1_m(j))} \circ \theta_e = \theta_j \circ \pi_{S^1_m(j)}$$

a.s. w.r.t. λ, so that $\theta_j \circ \pi_i = \theta_j \circ \pi_j$ a.s. w.r.t. $(S_m)_{**}\lambda$ for every m. Passing to the limit (in the coupling topology!) we see that $\theta_j^{-1}(\Sigma)$ is $e \cup \{j\}$-insensitive w.r.t $\tilde{\mu}$.

In particular the function $\tilde{h}$ is $\vee_{j\in e'}\tilde{\Sigma}_{e\cup\{j\}}$-measurable. By satedness of μ we obtain

$$\kappa = \int_{\tilde{X}} f \circ \theta_e \tilde{h} \mathrm{d}\tilde{\mu}^0 = \int_{\tilde{X}} f \circ \theta_e \mathbb{E}(\tilde{h}|\theta_e^{-1}(\vee_{j\in e'}\Sigma_{e\cup\{j\}}))\mathrm{d}\tilde{\mu}^0.$$

The latter expectation is a lift under θ_e of a $\vee_{j\in e'}\Sigma_{e\cup\{j\}}$-measurable function on X that correlates with f. □

We are now in position to obtain relative independence for more sophisticated oblique copies.

Lemma 10. *Let μ be a fully sated s.s. law and $I, I', I'' \subset \binom{k}{\geq 1}$ be up-sets such that $I' = I'' \cup \{e\}$ and $e \notin I$.*

For minimal elements $a \in I \cup I'$ let $\Xi_a \subset \Sigma_a$ be arbitrary sub-σ-algebras and let also $\Xi_a = \Sigma_a$ for other $a \in \binom{k}{\geq 1}$. Define $\Xi_e^\dagger$ and $\Xi_J^\dagger$ analogously to $\Sigma_e^\dagger$ and $\Sigma_J^\dagger$, respectively.

Then $\Sigma_I^\dagger$ and $\Xi_{I'}^\dagger$ are relatively independent over $\Xi_{I''}^\dagger$ under μ^1.

Proof. The conclusion is trivial if $e \in I''$, so assume $e \notin I''$. Then for every $a \in I \cup I''$ there exists $j(a) \in a \setminus e$.

Given $f \in L^2(\Sigma_I^\dagger) \subset L^2(\vee_{a\in I}\Sigma_{\{j(a)\}}^\dagger)$ such that $f \not\perp L^2(\Xi_{I'}^\dagger)$ we have to show $f \not\perp L^2(\Xi_{I''}^\dagger)$.

By the assumption there exist functions $h \in L^\infty(\Xi_{I''}^\dagger) \subset L^\infty(\vee_{a\in I''}\Sigma_{\{j(a)\}}^\dagger)$ and $g \in L^\infty(\Xi_e^\dagger) \subset L^\infty(\Sigma_e^\dagger)$ such that

$$0 \neq \kappa = \int fgh \mathrm{d}\mu^1.$$

Now the product fh is $\vee_{a\in I\cup I''}\Sigma_{\{j(a)\}}^\dagger$-measurable, hence

$$\kappa = \int fh\mathbb{E}(g|\vee_{a\in I\cup I''}\Sigma_{\{j(a)\}}^\dagger)\mathrm{d}\mu^1.$$

By Proposition 8 with $e' = j(I \cup I'')$ this expectation is measurable w.r.t. the σ-algebra $\vee_{a\in I\cup I''}\Sigma_{e\cup\{j(a)\}}^\dagger \subset \Xi_{I''}^\dagger$, so that $f \not\perp L^2(\Xi_{I''}^\dagger)$ as required. □

Lemma 11. *Let $B, B', B^\cap \subset B''$ be σ-algebras such that B and B' are relatively independent over B'' and B and B'' are relatively independent over $B^\cap$. Then B and B' are relatively independent over $B^\cap$.*

Proof. Let $f \in L^1(B')$. Then

$$\mathbb{E}(f|B \vee B^\cap) = \mathbb{E}(\mathbb{E}(f|B \vee B'')|B \vee B^\cap) = \mathbb{E}(\mathbb{E}(f|B'')|B \vee B^\cap) = \mathbb{E}(\mathbb{E}(f|B'')|B^\cap) = \mathbb{E}(f|B^\cap),$$

where we have used inclusion, relative independence, relative independence and inclusion, respectively. □

Theorem 12. *Let μ be a fully sated s.s. law and $I, I' \subset \binom{k}{\geq 1}$ be two up-sets. Then $\Sigma_I^\dagger$ and $\Sigma_{I'}^\dagger$ are relatively independent over $\Sigma_{I\cap I'}^\dagger$ under μ^1.*

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Proof. By induction on the size of $I\Delta I''$ using Lemma 10 with $\Xi_a = \Sigma_a$ and Lemma 11. $\square$

Theorem 13. *Let μ be a fully sated s.s. law and $I \subset I' \subset \binom{k}{\geq d}$ be up-sets.*
For $e \subset d$ with $|e| = d$ let $\Xi_e \subset \Sigma_e$ be arbitrary sub-σ-algebras and let also $\Xi_e = \Sigma_e$ if $|e| > d$. Then $\Sigma_I^\dagger$ and $\Xi_{I'}^\dagger$ are relatively independent over $\Xi_I^\dagger$ under μ^1.

Proof. By induction on the size of $I' \setminus I$ using Lemma 10 and Lemma 11. $\square$

1.6 The infinitary removal lemma

Theorem 14. *Let μ be a fully sated s.s. law and $1 \leq d \leq k$. Then for any positive functions $f_i \in L^\infty(\Sigma_{I_{i,d}})$, $i \in k$ where $I_{i,d} := \langle i\rangle \cap \binom{k}{\geq d}$ we have*

$$\int_{X^k} \prod_{i\in k} f_i \circ \pi_i d\mu^1 = 0 \quad\implies\quad \int_X \prod_{i\in k} f_i d\mu^0 = 0.$$

Proof. We use descending induction on d. The case $d = k$ is clear. Assume that the statement is known for $d+1$ and let $f_{i\in k} \in L^\infty(\Sigma_{I_{i,d}})$ be as in the hypothesis. Replacing f_i by the characteristic function of its support we may assume $f_i = 1_{A_i}$ with $A_i \in \Sigma_{I_{i,d}}$.

For each $e \in \binom{k}{d}$ let $(\Xi_{e,n})_n$ be an increasing sequence of finite σ-algebras that together generate Σ_e and for $e \in \binom{d}{\geq d+1}$ let $\Xi_{e,n} = \Sigma_e$. Let $\delta > 0$ be chosen later and $B_{i,n} := \{\mathbb{E}(A_i|\Xi_{I_{i,d},n}) > 1-\delta\}$, so that $B_{i,n} \to A_i$ (in L^1, say). Here and later we identify sets with their characteristic functions.

By definition of the algebra $\Xi_{I_{i,d},n}$ each set $B_{i,n} \in \Xi_{I_{i,d},n}$ can be written as a finite union of sets of the form

$$\cap_{e\in\langle i\rangle\cap\binom{k}{d}} C_{i,e} \cap \tilde{A}_i$$

with $C_{i,e} \in \Xi_e$ and $\tilde{A}_i \in \Sigma_{I_{i,d+1}}$. Assume for the moment that $\mu^1(\cap_i B_{i,n} \circ \pi_i) = 0$, so that

$$0 = \int_{X^k} \prod_{i\in k} \prod_{e\in\langle i\rangle\cap\binom{k}{d}} C_{i,e} \circ \pi_i \cdot \tilde{A}_i \circ \pi_i \mathrm{d}\mu^1 = \int_{X^k} \prod_{e\in\binom{k}{d}} \prod_{i\in e} C_{i,e} \circ \pi_{j(e)} \cdot \prod_{i\in k} \tilde{A}_i \circ \pi_i \mathrm{d}\mu^1,$$

where $j(e) \in e$ is arbitrary, and by Theorem 12 we obtain

$$0 = \int_{X^k} \prod_{e\in\langle i\rangle\cap\binom{k}{d}} \mathbb{E}(\prod_{i\in e} C_{i,e}|\Sigma_{\langle e\rangle\cap\binom{k}{\geq d+1}}) \circ \pi_{j(e)} \cdot \prod_{i\in k} \tilde{A}_i \circ \pi_i \mathrm{d}\mu^1.$$

Here we may replace the function $\mathbb{E}(\prod_{i\in e} C_{i,e}|\Sigma_{\langle e\rangle\cap\binom{k}{\geq d+1}})$ by its support that is a set in $\Sigma_{\langle e\rangle\cap\binom{k}{\geq d+1}} \subset \Sigma_{I_{j(e),d+1}}$. The induction hypothesis implies

$$0 = \int_X \prod_{e\in\cap\binom{k}{d}} \operatorname{supp}\mathbb{E}(\prod_{i\in e} C_{i,e}|\Sigma_{\langle e\rangle\cap\binom{k}{\geq d+1}}) \cdot \prod_{i\in k} \tilde{A}_i \mathrm{d}\mu^0 \geq \int_X \prod_{e\in\binom{k}{d}} \prod_{i\in e} C_{i,e} \cdot \prod_{i\in k} \tilde{A}_i \mathrm{d}\mu^0$$

since $\operatorname{supp}\mathbb{E}(\prod_{i\in e} C_{i,e}|\Sigma_{\langle e\rangle\cap\binom{k}{\geq d+1}}) \supseteq \prod_{i\in e} C_{i,e}$. Therefore $\mu^0(\cap_i B_{i,n})$ vanishes, being a sum of finitely many such terms, and we obtain

$$\mu^0(\cap_i A_i) = \lim_n \mu^0(\cap_i B_{i,n}) = 0.$$

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It remains to be seen that the set $F_n := \prod_i B_{i,n} \circ \pi_i$ is μ^1-null. For every i we have

$$\mu^1(F_n \setminus \pi_i^{-1}(A_i)) = \int_{X^k} (B_{i,n} \setminus A_i) \circ \pi_i \prod_{i' \neq i} B_{i',n} \circ \pi_{i'} = \int_{X^k} \mathbb{E}(B_{i,n} \setminus A_i | \Xi_{I_{i,d},n}) \circ \pi_i \prod_{i' \neq i} B_{i',n} \circ \pi_{i'}$$

since $\Sigma^\dagger_{I_{i,d}}$ and $\Xi^\dagger_{\binom{k}{\geq d},n}$ are relatively independent over $\Xi^\dagger_{I_{i,d},n}$ by Theorem 13. Note that

$$\mathbb{E}(B_{i,n} \setminus A_i | \Xi_{I_{i,d},n}) = B_{i,n}(1 - \mathbb{E}(A_i | \Xi_{I_{i,d},n})) \leq \delta B_{i,n},$$

so that

$$\mu^1(F_n \setminus \pi_i^{-1}(A_i)) \leq \delta \mu^1(F_n).$$

Therefore

$$\mu^1(F_n) \leq \mu^1(\cap_{i \in k} \pi_i^{-1}(A_i)) + \sum_{i \in k} \mu^1(F_n \setminus \pi_i^{-1}(A_i)) \leq 0 + \sum_{i \in k} \delta \mu^1(F_n).$$

If $\delta < 1/k$ this implies $\mu^1(F_n) = 0$. □

Corollary 15. *Let μ be a law on $\{0,1\}$ (with discrete topology) such that $\mu_n(\pi_w^{-1}(1)) \geq \delta > 0$ for every n and every word w of length n. Then there exists an n and a combinatorial line $S : k^1 \to k^n$ such that $\mu_n(\cap_{i \in k} \pi_{S(i)}^{-1}(1)) > 0$.*

Proof. Let $(S_m)_m$ be the sequence of combinatorial subspaces given by Lemma 1. It suffices to obtain the conclusion with S being the identity subspace $S : k^1 \to k^1$ and some law $(S_m)_{**}\mu$.

Since $\{1\}$ is clopen, it suffices to obtain the conclusion for the s.s. law given by Lemma 1, thus we may assume that μ is strongly stationary.

Clearly it suffices to obtain the conclusion for any extension of μ, so by Proposition 5 we may assume that μ is also fully sated. Now the result follows from Theorem 14 with $d = 1$. □

The density Hales-Jewett theorem follows by [FK91, Proposition 2.1].

Polynomials

The obvious polynomial generalization of the proof of Proposition 8 fails because one cannot conclude

$$\kappa = \int_{X^{k^n}} f(x_{r_{e,i}(w)}) \prod_{j \in e'} h_j(x_{r_{e,j}(w)}) \mathrm{d}\mu^n(x)$$

since the points $r_{e,j}(w)$ in general do not lie on a combinatorial line in k^{n^d}.

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