Dr. E. Hellmann WS 2013/14

Algebraic Geometry I

Exercise Sheet 10

Due Date: 09.01.2014

Exercise 1:

Let k be a field and $n, m \geq 1$.

(i) Show that there is a morphism $\tilde{f} : \mathbb{A}_k^{n+1} \times \mathbb{A}_k^{m+1} \to \mathbb{A}^{nm+m+n+1}$ that is on R-valued points given by

$$(a_0, \dots, a_n), (b_0, \dots, b_m) \longmapsto (a_i b_j)_{\substack{i=0,\dots,n \\ j=0,\dots,m}}$$

for an k-algebra R and describe the induced morphism of k-algebras on the global sections of the structure sheaves.

(ii) Given an integer l write $X_l \subset \mathbb{A}_k^l$ for the open subscheme $X_l = \mathbb{A}_k^l \backslash \{0\}$. Show that $\tilde{f}$ induces a morphism $X_{n+1} \times X_{m+1} \to X_{nm+n+m+1}$.

(iii) For an integer l let $p_l : X_{l+1} \to \mathbb{P}_k^l$ denote the canonical projection. Show that there is a unique morphism $f : \mathbb{P}_k^n \times \mathbb{P}_k^m \to \mathbb{P}_k^{nm+n+m}$ such that the following diagram commutes:

$$\begin{array}{ccc} X_{n+1} \times X_{m+1} & \xrightarrow{\tilde{f}} & X_{nm+n+m+1} \\ \downarrow{\scriptstyle p_n \times p_m} & & \downarrow{\scriptstyle p_{nm+n+m}} \\ \mathbb{P}_k^n \times \mathbb{P}_k^m & \xrightarrow[f]{} & \mathbb{P}_k^{nm+n+m} \end{array}$$

(iv) Show that f is a closed immersion called the *Segre embedding*.

(v) Describe the image of the Segre embedding $\mathbb{P}_k^1 \times \mathbb{P}_k^1 \hookrightarrow \mathbb{P}_k^3$.

Exercise 2:

For an $\mathbb{R}$-scheme X write $X_{\mathbb{C}}$ for the extension of scalars from $\mathbb{R}$ to $\mathbb{C}$ and $\sigma_X : X_{\mathbb{C}} \to X_{\mathbb{C}}$ for the automorphism of $X_{\mathbb{C}}$ induced by the complex conjugation on $\mathbb{C}$.

(i) Let X and Y be $\mathbb{R}$-schemes. Show that a morphism $f : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ of $\mathbb{C}$-schemes is the extension of scalars of a morphism of $\mathbb{R}$-schemes $f_0 : X \to Y$ if and only if $f \circ \sigma_X = \sigma_Y \circ f$ in which case f_0 is uniquely determined.

(ii) Let X be an $\mathbb{R}$-scheme such that $X_{\mathbb{C}} \cong \mathbb{A}_{\mathbb{C}}^1$. Show that $X \cong \mathbb{A}_{\mathbb{R}}^1$.

(iii) Let X be an $\mathbb{R}$-scheme such that $X_{\mathbb{C}} \cong \mathbb{P}_{\mathbb{C}}^1$. Show that either $X \cong V_+(T_0^2 + T_1^2 + T_2^2) \subset \mathbb{P}_{\mathbb{R}}^2$ or $X \cong \mathbb{P}_{\mathbb{R}}^1$.

Exercise 3:

(i) Let $X = \operatorname{Spec} A$ be an integral k-scheme of finite type and let $0 \neq f \in A$. Show that the closed subscheme $\operatorname{Spec} A/(f)$ of X has dimension $\dim X - 1$.
(Hint: Krull's principal ideal theorem)

(ii) Let $Y = \operatorname{Spec} k[T_1, T_2]/(T_1T_2, T_1^2) \subset \operatorname{Spec} k[T_1, T_2] = \mathbb{A}_k^2$. Show that Y is irreducible and $\dim Y = 1 = \dim \mathbb{A}_k^2 - 1$ but there is no $f \in k[T_1, T_2]$ such that $Y = V(f)$.

(iii) Let X be a k-scheme of finite type and $x \in X$. Show that $\dim \overline{\{x\}}$ equals the trancendence degree of $\kappa(x)$ over k.

Exercise 4:

Let S be a base scheme and let $p : G \to S$ be an S-scheme such that the functor $G : \mathrm{Sch}_S^{\mathrm{opp}} \to (\mathrm{Sets})$ factors through the forgetful functor $(\mathrm{Groups}) \to (\mathrm{Sets})$. Show that G is a group scheme over S i.e. there exist morphisms $m : G \times_S G \to G$ and $i : G \to G$ as well as a section $e : S \to G$ to p such that the following diagrams are commutative:

Associativity:

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{\mathrm{id}_G \times m} & G \times_S G \\ {\scriptstyle m \times \mathrm{id}_G}\downarrow & & \downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow[m]{} & G \end{array}$$

Existence of neutral element:

$$\begin{array}{ccc} G & \xrightarrow{(e \circ p, \mathrm{id}_G)} & G \times_S G \\ {\scriptstyle (\mathrm{id}_G, e \circ p)}\downarrow & \searrow^{\mathrm{id}_G} & \downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow[m]{} & G \end{array}$$

Existence of inverse elements:

$$\begin{array}{ccc} G & \xrightarrow{(i, \mathrm{id}_G)} & G \times_S G \\ {\scriptstyle (\mathrm{id}_G, i)}\downarrow & \searrow^{e \circ p} & \downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow[m]{} & G \end{array}$$

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeom`

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