Dr. E. Hellmann WS 2013/14

Algebraic Geometry I

Exercise Sheet 2

Due Date: 31.10.2013

Exercise 1:

Let X, Y, Z be affine algebraic sets and let $f : X \to Y$ and $g : Y \to Z$ be morphisms of affine algebraic sets.

(i) Show that $g \circ f : X \to Z$ is a morphism of affine algebraic sets.

(ii) Show that $(g \circ f)^\sharp = f^\sharp \circ g^\sharp : \mathcal{O}(Z) \to \mathcal{O}(X)$.

(iii) Let $y \in Y$. Show that the fiber $f^{-1}(y)$ is an affine algebraic set and that $\mathcal{O}(f^{-1}(y))$ agrees with the quotient of $\mathcal{O}(X) \otimes_{\mathcal{O}(Y)} (\mathcal{O}(Y)/\mathfrak{m}_y)$ by its nilradical.

Exercise 2:

(i) Let $Z_1 = V(T_2(T_1 - 1) - 1)$ and $Z_2 = V(T_2^2 - T_1^2(T_1 + 1))$ be affine algebraic subsets of $\mathbb{A}^2$. Show that $(t_1, t_2) \mapsto (t_1^2 - 1, t_1(t_1^2 - 1))$ induces a bijective morphism $f : Z_1 \to Z_2$ which is not an isomorphism.

(ii) Show that $t \mapsto (t^2, t^3)$ defines a morphism $g : \mathbb{A}^1 \to V(T_2^2 - T_1^3) \subset \mathbb{A}^2$ which is a homeomorphism but not an isomorphism.

Exercise 3:

Let X be an affine algebraic set and let $x \in X$. Show that the local ring $\mathcal{O}_{X,x}$ is a domain if and only if there is a unique irreducible component Z of X containing x.

Let $\mathcal{C}$ be a category and I a partially ordered index set.

An *inductive system* $(X_i, f_{ij})_{i,j\in I}$ in $\mathcal{C}$ is a collection of objects X_i of $\mathcal{C}$ for each $i \in I$ and for each $i \leq j$ in I a morphism $f_{ij} : X_i \to X_j$ such that $f_{ii} = \mathrm{id}_{X_i}$ and $f_{jk} \circ f_{ij} = f_{ik}$ for all $i \leq j \leq k$.

An object X of $\mathcal{C}$ together with morphisms $f_i : X_i \to X$ for $i \in I$ satisfying $f_j \circ f_{ij} = f_i$ for all $i \leq j$ is called the *colimit* of the system (X_i, f_{ij}) if it satisfies the following universal property: For any object Y of $\mathcal{C}$ together with morphisms $g_i : X_i \to Y$ for $i \in I$ satisfying $g_j \circ f_{ij} = g_i$ for all $i \leq j$ there exists a unique morphism $g : X \to Y$ such that $g \circ f_i = g_i$ for all $i \in I$, i.e. such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, dashed, "g"] & & Y \\
& X_j \arrow[ul, "f_j"'] \arrow[ur, "g_j"] & \\
& X_i. \arrow[u, "f_{ij}"] \arrow[uul, bend left, "f_i"] \arrow[uur, bend right, "g_i"'] &
\end{tikzcd}$$

In this case we write $X = \varinjlim_{i\in I} X_i$.

If I has the property that for all $i, j \in I$ there exists some $k \in I$ such that $i \leq k$ and $j \leq k$, then a system $(X_i, f_{ij})_{i,j\in I}$ is called a *direct system* (or *filtered inductive system*).
If it exists, the colimit $\varinjlim_{i\in I} X_i$ of a direct system is called a *direct limit* (or *filtered colimit*).

Exercise 4:

(i) Let $J \subset I$ be a cofinal subset and let $(X_i, f_{ij})_{i,j\in I}$ be a given direct system in $\mathcal{C}$ such that $\varinjlim_{i\in I} X_i$ exists in $\mathcal{C}$. Show that $\varinjlim_{j\in J} X_j$ exists in $\mathcal{C}$ and that there is a canonical isomorphism

$$\varinjlim_{j\in J} X_j \longrightarrow \varinjlim_{i\in I} X_i.$$

(ii) Let A be commutative ring. Show that direct limits exist in the category of A-algebras.

Hint: Given a direct system (A_i, f_{ij}), consider the quotient of $\coprod_{i\in I} A_i$ by the equivalence relation $A_i \ni a \sim b \in A_j$ if and only if $f_{ik}(a) = f_{jk}(b)$ for some $k \geq i, j$.

(ii)* Let A be a commutative ring. Show that arbitrary colimits exist in the category of A-algebras.

Hint: Given an inductive system $(A_i, f_{ij})_{i,j\in I}$, try to realize the colimit of the A_i as a quotient of the restricted tensor product $\bigotimes'_{i\in I} A_i$. The restricted tensor product itself is defined as

$$\bigotimes\nolimits'_{i\in I} A_i = \varinjlim_{J\subset I \text{ finite}} \bigotimes\nolimits_{i\in J} A_i.$$

(iii) Let A be a commutative ring and let $\mathfrak{p} \subset A$ be a prime ideal. Show that

$$A_{\mathfrak{p}} = \varinjlim_{f\notin\mathfrak{p}} A_f,$$

where $\{f \notin \mathfrak{p}\}$ is partially ordered by $f \leq g$ if and only if $g = fh$ for some $h \in A$ in which case $A_f \to A_g$ is the obvious localization map.

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeom`