Dr. E. Hellmann SS 2014

Algebraic Geometry II

Exercise Sheet 10

Due Date: 07.07.2014

Exercise 1:

In this exercise we prove Serre's criterion: Let X be a quasi-compact scheme such that $H^1(X, \mathscr{I}) = 0$ for all quasi-coherent sheaves of ideals $\mathscr{I}$. Then X is affine. To prove this, proceed as follows:

(i) Let $x \in X$ be a closed point and U an affine neighborhood of x. Let $\mathscr{I}_Y$ be the sheaf of ideals defining a scheme structure on the closed complement Y of U. Further let $\mathscr{I}_{Y \cup \{x\}}$ be the sheaf of ideals vanishing on $Y \cup \{x\}$. Use the short exact sequence

$$0 \longrightarrow \mathscr{I}_{Y \cup \{x\}} \longrightarrow \mathscr{I}_Y \longrightarrow \kappa(x) \longrightarrow 0$$

to show that there exists $f \in \Gamma(X, \mathcal{O}_X)$ with $f(x) \neq 0$ such that $X_f = \{y \in X \mid f(y) \neq 0\}$ is affine.
Hint: Show that there exists an f with $X_f \subset U$

(ii) Let $A = \Gamma(X, \mathcal{O}_X)$. Then by (i) there exist $f_1, \dots, f_n \in A$ such that $X = \bigcup X_{f_i}$. Show that $(f_1, \dots, f_n) = A$.
Hint: Let

$$\varphi : \bigoplus_{i=1}^{n} \mathcal{O}_X \longrightarrow \mathcal{O}_X$$

be the morphism $(s_i) \mapsto \sum f_i s_i$. Reduce the claim to the claim $H^1(X, \ker \varphi) = 0$. Then use a filtration of $\ker \varphi$ by quasi-coherent sheaves $\mathscr{F}_i$ such that $\mathscr{F}_i / \mathscr{F}_{i-1}$ is a sheaf of ideals on X to prove the vanishing of H^1.

(iii) Deduce that $X = \operatorname{Spec} A$.

Exercise 2:

Let X be a scheme and let $\mathscr{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals defining a closed immersion $Y \hookrightarrow X$. Assume that $\mathscr{I}$ is nilpotent. Show that X is affine if and only if Y is.

Exercise 3:

Let $X = V_+(f) \subset \mathbb{P}^2_k$ be a closed subscheme defined by some $f \in k[T_0, T_1, T_2]$ that is homogenous of degree d. Assume that $(1:0:0) \notin X$ (hence X can be covered by $U = D_+(T_1) \cap X$ and $V = D_+(T_2) \cap X$). Show that

$$\dim_k H^0(X, \mathcal{O}_X) = 1,$$
$$\dim_k H^1(X, \mathcal{O}_X) = \tfrac{1}{2}(d-1)(d-2),$$

by explicitly computing the Čech-complex of X.

Exercise 4:

Let X be an algebraically closed field and let $X = V(T_2(T_2 - T_1^2 + 1)) \subset \mathbb{A}^2_k$. Compute $H^i(X, \underline{\mathbb{Z}}_X)$.

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeomII`