Dr. E. Hellmann SS 2014

Algebraic Geometry II

Exercise Sheet 3

Due Date: 12.05.2014

Exercise 1:

(i) Let k be a field and let $1 \leq m < n$. Show that the Grassmann-variety $\mathrm{Gr}_{m,n}$ of m-dimensional subspaces in k^n is proper over k.

(ii) Let $f : \mathbb{A}^1_k \to \mathrm{Gr}_{2,4}$ be the morphism defined by the family of subspaces

$$\mathcal{O}_{\mathbb{A}^1_k}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \oplus \mathcal{O}_{\mathbb{A}^1_k}\begin{pmatrix}0\\T\\1\\0\end{pmatrix} \subset \mathcal{O}^4_{\mathbb{A}^1_k}$$

on $\mathbb{A}^1_k = \mathrm{Spec}\ k[T]$. Show that the morphism f extends to a unique morphism $\tilde{f} : \mathbb{P}^1_k \to \mathrm{Gr}_{2,4}$ and compute $\tilde{f}(\infty)$.

Exercise 2:

(i) Let $f : X \to Y$ be a morphism of finite type and assume that Y is locally noetherian. Show that in the valuative criteria for separatedness and properness it is enough to consider

(a) complete discrete valuation rings.

(b) discrete valuation rings with algebraically closed residue field.

(ii) Let X be a proper k-scheme and let C be curve over k, i.e. C is a 1-dimensional integral scheme of finite type over k. Let $P \in C$ be a closed point such that C is smooth at P (i.e. $\mathcal{O}_{C,P}$ is a discrete valuation ring). Let $f : C\backslash\{P\} \to X$ be a morphism of k-schemes. Show that there is a unique morphism $\tilde{f} : C \to X$ extending f.

(iii) Let $f : X \to Y$ be a morphism of finite type. For $i = 1, \dots, n$ let $X_i \subset X$ and $Y_i \subset Y$ be closed subschemes such that $f|_{X_i} \to Y$ factors over $Y_i \hookrightarrow Y$ and such that $X = \bigcup_{i=1}^n X_i$. Write $f_i : X_i \to Y_i$ for the induced morphisms. Show that f is proper if and only if all the f_i are proper.

Exercise 3:

Let Y be a locally noetherian scheme and let $\mathscr{S} = \bigoplus_{d\geq 0} \mathscr{S}$ be a quasi-coherent graded $\mathcal{O}_Y$-algebra such that $\mathscr{S}_1$ is coherent and $\mathscr{S}$ is generated by $\mathscr{S}_1$. Let $X = \underline{\mathrm{Proj}}_Y \mathscr{S}$ and write $p : X \to Y$ for the canonical projection to Y.

(i) Show that there is a functor $\mathscr{M} \mapsto \tilde{\mathscr{M}}$ from the category of quasi-coherent graded $\mathscr{S}$-modules to the category of quasi-coherent $\mathcal{O}_X$-modules, globalizing the construction in the case $Y = \mathrm{Spec}\ A$.

(ii) Let $\mathcal{O}(n)$ be the locally free $\mathcal{O}_X$-module associated to $\mathscr{S}[n]$ and, given a quasi-coherent sheaf $\mathscr{F}$ on X, define

$$\Gamma_*(\mathscr{F}) = \bigoplus_{d\in\mathbb{Z}} p_*(\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}(n)).$$

This defines a functor

$$\Gamma_* : \{\text{quasi-coherent } \mathcal{O}_X\text{-modules}\} \longrightarrow \{\text{quasi-coherent graded } \mathscr{S}\text{-modules}\}.$$

Show that $\Gamma_*(-)$ and $(-)\tilde{}$ are adjoint functors and that the canonical morphism

$$\Gamma_*(\mathscr{F})\tilde{} \longrightarrow \mathscr{F}$$

is an isomorphism for all quasi-coherent sheaves $\mathscr{F}$ on X.

Exercise 4:
Let k be a field and $X = \mathbb{P}^3_k = \operatorname{Proj} k[T_0, T_1, T_2, T_3]$. Let

$$C = V_+(T_2, T_3),\ D = V_+(T_0^2 - T_1^2 + T_0T_2, T_3) \subset X$$

be closed subschemes of dimension 1 with corresponding sheaves of ideals $\mathscr{I}_C, \mathscr{I}_D \subset \mathcal{O}_X$.

(i) Show that the (scheme-theoretic) intersection of C and D consists precisely of the points $P = (1 : 1 : 0 : 0)$ and $Q = (1 : -1 : 0 : 0)$.

(ii) Let $X_1 = \mathrm{Bl}_{C\backslash P} X\backslash P$ be the blow up of $X\backslash P$ in $C\backslash P$ and let $\pi_1 : X_1 \to X\backslash P$ be the canonical projection. Let $Z_1 \subset X_1$ be the closed subscheme defined by $\pi_1^{-1}(\mathscr{I}_D|_{X\backslash P})\mathcal{O}_{X_1} \subset \mathcal{O}_{X_1}$ and let X_1' be the blow up of X_1 in Z_1. Write $\pi_1' : X_1' \to X\backslash P$ for the canonical projection.

Let $X_2 = \mathrm{Bl}_{D\backslash Q} X\backslash Q$ be the blow up of $X\backslash Q$ in $D\backslash Q$ and let $\pi_2 : X_2 \to X\backslash D$ be the canonical projection. Let $Z_2 \subset X_2$ be the closed subscheme defined by $\pi_2^{-1}(\mathscr{I}_C|_{X\backslash D})\mathcal{O}_{X_2} \subset \mathcal{O}_{X_2}$ and let X_2' be the blow up of X_2 in Z_2. Write $\pi_2' : X_2' \to X\backslash Q$ for the canonical projection.

Show that there is a canonical isomorphism ${\pi_1'}^{-1}(X\backslash\{P, Q\}) \to {\pi'}_2^{-1}(X\backslash\{P, Q\})$.

(iii) Let Y be the scheme obtained by gluing X_1' and X_2' along the isomorphism

$${\pi_1'}^{-1}(X\backslash\{P, Q\}) \to {\pi'}_2^{-1}(X\backslash\{P, Q\}).$$

Show that the canonical morphism $f : Y \to X$ is proper.

(*In fact the morphism f is not projective. However, this is much harder to show.*)

(iv) Show that there is a closed subscheme $Z \subset Y$ such that the morphism $\mathrm{Bl}_Z Y \to X$ given by composing the canonical morphism $\mathrm{Bl}_Z Y \to Y$ with f is projective.

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeomII`