Equivariant bordism from the global perspective

Stefan Schwede

Mathematisches Institut, Universität Bonn

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Introduction

Global homotopy theory =

‘equivariant homotopy theory with maximal symmetry’

global = all compact Lie groups act compatibly

Aim:
- explain a rigorous formalism
- motivate the theory by a geometric example

I. Global stable homotopy theory

- Orthogonal spectra
- Global equivalences
- Examples

II. Global equivariant bordism

- Equivariant bordism
- Global Thom spectra

Orthogonal spectra

Definition

An orthogonal spectrum X consists of

- based $O(V)$-spaces $X(V)$, for every inner product space V
- $O(V) \times O(W)$-equivariant structure maps

$$\sigma_{V,W} \; : \; X(V) \wedge S^W \longrightarrow X(V \oplus W)$$

subject to associativity and identity conditions.

Here: $S^W = W \cup \{\infty\}$ one-point compactification

An orthogonal spectrum X has an underlying spectrum in the sense of stable homotopy theory:

- $X_n = X(\mathbb{R}^n), \quad n \geq 0$
- $\sigma_{\mathbb{R}^n,\mathbb{R}} : \Sigma X_n = X(\mathbb{R}^n) \wedge S^1 \longrightarrow X(\mathbb{R}^{n+1}) = X_{n+1}$
- forget the $O(n)$-actions

Equivariant homotopy groups

Let X be an orthogonal spectrum.

- G: compact Lie group
- V: orthogonal G-representation

$\Big\} \Longrightarrow$ G acts on $X(V)$

$[S^V, X(V)]^G$: based G-homotopy classes of G-maps

Definition

The *G-equivariant stable homotopy group* of X is

$$\pi_0^G(X) \ = \ \mathrm{colim}_V \, [S^V, X(V)]^G \, .$$

- colimit by stabilization via $- \wedge S^W$, using structure maps
- $\pi_0^G(X)$ is an abelian group, natural in X
- similarly: $\pi_k^G(X)$ for $k \in \mathbb{Z}$

Global equivalences

Definition

A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a global equivalence if the map

$$\pi_k^G(f) \ : \ \pi_k^G(X) \ \longrightarrow \ \pi_k^G(Y)$$

is an isomorphism for all $k \in \mathbb{Z}$ and all G.

Definition

The global stable homotopy category is

$$\mathcal{GH} \ = \ \mathrm{Sp}^O[\text{global equivalences}^{-1}] \ ,$$

the localization of orthogonal spectra at the class of global equivalences.

Global stable homotopy category

- Model category structures are available
- $\mathcal{GH}$ is a tensor triangulated category
- objects in $\mathcal{GH}$ represent cohomology theories on stacks (Gepner-Henriques, Gepner-Nikolaus)

Note: $\pi_k^{\{e\}}(X)$ = traditional (non-equivariant) homotopy group of the underlying spectrum of X, so

$$\text{global equivalence} \implies \text{stable equivalence}$$

The forgetful functor

$$\mathcal{GH} \longrightarrow \text{(stable homotopy category)}$$

has fully faithful adjoints providing a recollement.

Restriction and transfers

A continuous homomorphism $G \longleftarrow K : \alpha$
induces a restriction homomorphism $\alpha^* : \pi_0^G(X) \longrightarrow \pi_0^K(X)$

$$[f : S^V \longrightarrow X(V)] \longmapsto [\alpha^*(f) : S^{\alpha^*(V)} \longrightarrow X(\alpha^*(V))]$$

A closed subgroup $H \leq G$ gives rise to
a transfer homomorphism $\mathrm{tr}_H^G : \pi_0^H(X) \longrightarrow \pi_0^G(X)$
(equivariant Thom-Pontryagin construction)

Relations:

- restrictions are contravariantly functorial
- transfers are covariantly functorial
- inner automorphisms are identity
- transfers commute with inflation
- double coset formula

$\Longrightarrow$ 'global functors' ('inflation functors')

Examples

Example

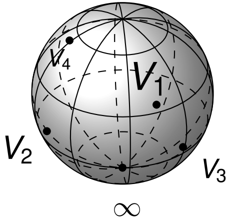
The global sphere spectrum $\mathbb{S}$ is given by

$$\mathbb{S}(V) = S^V , \qquad \sigma_{V,W} : S^V \wedge S^W \cong S^{V \oplus W}$$

Example

The connective global K-theory spectrum **ko**:

$\mathbf{ko}(V) =$ finite configurations of points in S^V labeled by finite dimensional orthogonal subspaces of $\mathrm{Sym}(V)$

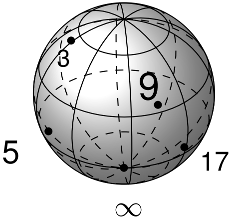


Example

The Eilenberg-Mac Lane spectrum $H\mathbb{Z}$:

$$(H\mathbb{Z})(V) = Sp^{\infty}(S^V)$$

infinite symmetric product



Some global morphisms

For G finite:

$\mathbb{S}$	$\pi_0^G(\mathbb{S}) = A(G)$	Burnside ring (Segal)
$\downarrow$ label by $\mathbb{R}\cdot 1$	$\downarrow$ permutation representation	
ko	$\pi_0^G(\mathbf{ko}) = RO(G)$	representation ring
$\downarrow$ dimension	$\downarrow$ rank	
$H\mathbb{Z}$	$\pi_0^G(H\mathbb{Z}) = \mathbb{Z}$	constant global functor

Global versus non-equivariant equivalence:

- The morphism $\mathbb{S}_{\mathbb{Q}} \longrightarrow H\mathbb{Q}$ is a non-equivariant equivalence, but not a global equivalence.
- The morphism $\mathbf{mO} \longrightarrow \mathbf{MO}$ is a non-equivariant equivalence, but not a global equivalence.

Equivariant bordism

G: compact Lie group, X: G-space

Definition

$\mathcal{N}_n^G(X) =$ G-equivariant bordism group of X
elements: bordism classes of (M, h) with:

- M: smooth closed G-manifold of dimension n
- $h : M \longrightarrow X$: continuous G-map

$\mathcal{N}_n^G(-)$ is covariant functor, abelian group by disjoint union

Equivariant homology theory (Conner-Floyd, Stong,...):

- G-homotopy invariant
- $\bigoplus \mathcal{N}_k^G(X_i) \xrightarrow{\cong} \mathcal{N}_k^G(\coprod X_i)$
- a G-map $f : X \longrightarrow Y$ yields a long exact sequence

$$\dots \longrightarrow \mathcal{N}_n^G(X) \xrightarrow{f_*} \mathcal{N}_n^G(Y) \xrightarrow{i_*} \widetilde{\mathcal{N}}_n^G(\mathrm{Cone}(f)) \xrightarrow{\partial} \mathcal{N}_{n-1}^G(X) \longrightarrow \dots$$

Examples

Non-equivariant bordism:

$$\mathcal{N}_* = \mathbb{F}_2[x_i \mid i \neq 2^n - 1]$$

Possible generators: $x_i = [\mathbb{R}P^i]$, i even; $x_i = [S^m \times_\tau \mathbb{C}P^n]$, i odd

Bordism of manifolds with involution:

- Define $\Gamma : \mathcal{N}_k^{C_2} \longrightarrow \mathcal{N}_{k+1}^{C_2}$ by $\Gamma[M, \tau] = S^1 \times_\tau M, \quad (z, x) \simeq (-z, \tau(x))$ with involution $[z, x] \mapsto [-\bar{z}, x]$.
- Set $y_k = [\mathbb{R}P^k, \tau]$, $[x_0 : x_1 : \ldots : x_n] \mapsto [-x_0 : x_1 : \ldots : x_n]$.

Then $\mathcal{N}_*^{C_2}$ is a free $\mathcal{N}_*$-module with basis

$$1, \quad \Gamma^n(y_{k_1} \cdot \ldots \cdot y_{k_r})$$

for $n \geq 0, r \geq 1, k_i \geq 2$.

$\mathbb{R}P^2$, $\tau =?$

Bordism and Thom spectra

Theorem (Thom '54)

Non-equivariant bordism is represented by a spectrum MO:

$$\mathcal{N}_n(X) \;\cong\; \mathrm{colim}_k\, [S^{n+k}, MO_k \wedge X_+]$$

nowadays: Thom spectrum and
Thom-Pontryagin construction

Thom: version for oriented bordism (*MSO*)

also: almost complex (*MU*), spin (*MSpin*), ...

René Thom

Questions:

- *G*-equivariant version?
- Global version?

Global Thom spectra

V: inner product space of dimension n
γ_V: tautological n-plane bundle
over the Grassmannian $Gr_n(V \oplus \mathbb{R}^\infty)$

Definition

The global Thom spectrum **mO** is the orthogonal spectrum with

$$\mathbf{mO}(V) = \text{Thom space of } \gamma_V \ .$$

The action of $O(V)$ and structure maps only affect V, not $\mathbb{R}^\infty$.

Small changes can make a big difference:

- replacing $Gr_n(V \oplus \mathbb{R}^\infty)$ by $Gr_n(V \oplus V)$ yields an orthogonal Thom spectrum **MO** with different equivariant homotopy types.
- **mO** is equivariant connective; **MO** is equivariantly oriented

Equivariant Thom-Pontryagin construction

Smooth compact G-manifolds can be embedded into G-representations (Mostow-Palais), so the equivariant Thom-Pontryagin construction makes sense:

$$\mathcal{N}_n^G(X) \longrightarrow \text{colim}_V [S^{V \oplus \mathbb{R}^n}, \mathbf{mO}(V) \wedge X_+] = \mathbf{mO}_n^G(X)$$

Theorem (Wasserman '69)

Let G be isomorphic to the product of a finite group and a torus. Then the equivariant Thom-Pontryagin construction is an isomorphism of equivariant homology theories.

The equivariant Thom-Pontryagin construction is not in general bijective. For example, the map

$$\mathcal{N}_0^{SU(2)} \longrightarrow \pi_0^{SU(2)}(\mathbf{mO})$$

is not surjective.

Induction versus transfer

Question:
Why finite×torus? What goes wrong in general?
A closer look at the functoriality for closed subgroups $H \leq G$

Geometry:
induction isomorphism:

$$\mathcal{N}^H_{n-d}(X) \xrightarrow{\mathrm{Ind}^G_H} \mathcal{N}^G_n(G \times_H X)$$
$$[M, h] \longmapsto [G \times_H M, G \times_H h]$$

where $d = \dim(G/H)$
→ shift by dimension

Homotopy theory:
'Wirthmüller isomorphism':

$$\mathbf{mO}^H_n(S^L \wedge X_+) \xrightarrow{\mathrm{Tr}^G_H} \mathbf{mO}^G_n(G \times_H X_+)$$

where $L = T_H(G/H)$
→ twist by an H-representation

Answer:
Different formal behaviour of induction / transfer.
So no chance for an isomorphism in general.

Why 'finite×torus' !

However:

G is isomorphic to the product of a finite group and a torus

$\Longleftrightarrow$ for every closed subgroup H of G
the tangent H-representation $T_H(G/H)$ is trivial

$\Longleftrightarrow$ all transfers 'up to G' are untwisted

In fact, this suggests a homotopy theoretic proof
(induction over the size of G, isotropy separation)

More refined statement: let V be a G-representation
$p : S(V \oplus \mathbb{R}) \longrightarrow S^V$ stereographic projection
represents a tautological equivariant bordism class

$$d_{G,V} \in \widetilde{\mathcal{N}}^G_{|V|}(S^V)$$

Correction by tautological class

Recall: $L = T_H(G/H)$ tangent H-representation,
of dimension $d = \dim(G/H)$

Proposition

For every closed subgroup H of a compact Lie group G and every H-space X the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{N}^H_{n-d}(X) & \xrightarrow{TP} & \mathbf{mO}^H_{n-d}(X_+) \\
 & & \big\downarrow d_{H,L} \times - \\
Ind^G_H \big\downarrow \cong & & \mathbf{mO}^H_n(S^L \wedge X_+) \\
 & & \cong \big\downarrow Tr^G_H \\
\mathcal{N}^G_n(G \times_H X) & \xrightarrow[TP]{} & \mathbf{mO}^G_n((G \times_H X)_+)
\end{array}$$

- the tautological class $d_{H,L}$ measures the failure of Thom-Pontryagin map to commute with induction/transfer.

Stable equivariant bordism and **MO**

- The classes $d_{G,V}$ are not invertible in $\mathcal{N}_*^G(-)$ nor $\mathbf{mO}_*^G(-)$.
- Formally inverting them forces

  'geometric induction = homotopical transfer'.

Corollary (Bröcker-Hook '72)

After formally inverting all tautological classes in $\mathcal{N}_*^G(-)$ and in $\mathbf{mO}_*^G(-)$, the Thom-Pontryagin construction becomes an isomorphism for all compact Lie groups G and all G-spaces X.

Formally inverting the classes $d_{G,V}$ yields:

- stable equivariant bordism:

$$\mathfrak{N}_n^{G:S}(X) \;=\; \operatorname{colim}_V \; \widetilde{\mathcal{N}}_{n+|V|}^G(S^V \wedge X_+)$$

- tom Dieck's homotopical equivariant bordism:

$$\mathbf{MO}_n^G(X) \;=\; \operatorname{colim}_V \; \mathbf{mO}_{n+|V|}^G(S^V \wedge X_+)$$

Summary

Open questions:

- Does $\mathbf{mO}_*^G(-)$ describe any geometric G-bordism theory? We need to twist induction by the tangent representation...
- Are there generalizations to equivariant bordism theories with more structure ($\mathbf{mSO}_*^G$, $\mathbf{mSpin}_*^G$, $\mathbf{mU}_*^G$,...)? Induction needs extra structure on G/H ...

Summary:

- The global stable homotopy category is the home of all equivariant phenomena with 'maximal symmetry'
- Orthogonal spectra and global equivalences provide a convenient model
- The global perspective reveals the difference between geometric bordism and equivariant Thom spectra

Reference: S. Schwede, *Global homotopy theory*

`www.math.uni-bonn.de/people/schwede/global.pdf`

Preview

Preview to Part II:

A global description of **mO** (analogues for **mSO**, **mU**,...):

- $\mathbf{mO} = \mathrm{hocolim}_m \, \mathbf{mO}_{(m)}$, where
  $\mathbf{mO}_{(m)}$ is a specific global refinement of $\Sigma^m MT(m)$
- exact triangles in the global stable homotopy category:

$$S^{m-1} \wedge B_{\mathrm{gl}} O(m) \longrightarrow \mathbf{mO}_{(m-1)} \longrightarrow \mathbf{mO}_{(m)} \longrightarrow S^m \wedge B_{\mathrm{gl}} O(m)$$

- Universal property: **mO** is obtained from $\mathbb{S}$
  by inductively coning off the classes

$$\mathrm{Tr}_{O(m-1)}^{O(m)}(d_{O(m-1),\mathbb{R}^{m-1}})$$

- This generalizes : $\mathrm{Tr}_{\{e\}}^{O(1)}(1) = 0$