Recent progress on the formal degree conjecture

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Outline

$\underline{G}$: split semisimple $\mathbb{Q}_p$-group, e.g.

$$\underline{G} = \mathrm{SL}_n, \mathrm{Sp}_{2n}, \mathrm{SO}_n, \mathrm{Aut}(\text{octonions}), \dots$$

$G \stackrel{\text{def}}{=} \underline{G}(\mathbb{Q}_p)$ (locally compact group)

1. Formal degree
2. Yu's supercuspidals
3. Formal degree conjecture
4. Gross and Reeder's refinement

Part 1. Formal degree

H: topological group.

$$\mathrm{Spec}_{\mathrm{u}}\, H \stackrel{\text{def}}{=} \frac{\{\text{irred. unitary reps of } H\}}{\text{iso.}} \qquad (\text{unitary dual})$$

Admits a (Fell) topology and (Plancherel) measure.

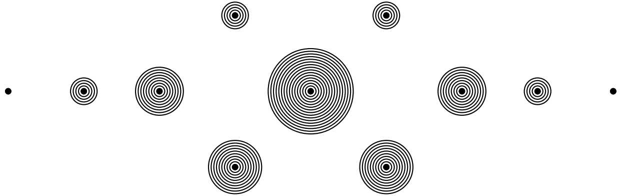
Natural questions:

1. What is $\mathrm{Spec}_{\mathrm{u}}\, H$ as a set?
2. What is the topology and measure on $\mathrm{Spec}_{\mathrm{u}}\, H$?

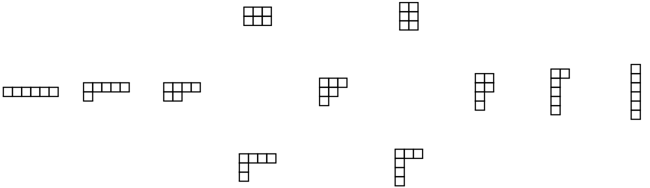
Toy model: Symmetric group

$$\mathrm{Spec}_{\mathrm{u}}\, S_n \longleftrightarrow \{\text{partitions of } n\}$$

$\mathrm{Spec}_{\mathrm{u}}\, S_6$



$\lambda \vdash 6$



Toy model: Symmetric group (dimension formula)

Goal: understand irreps in terms of their "parameters":

dim, dual, $\otimes$, Hom, Ind, Res, character, ...

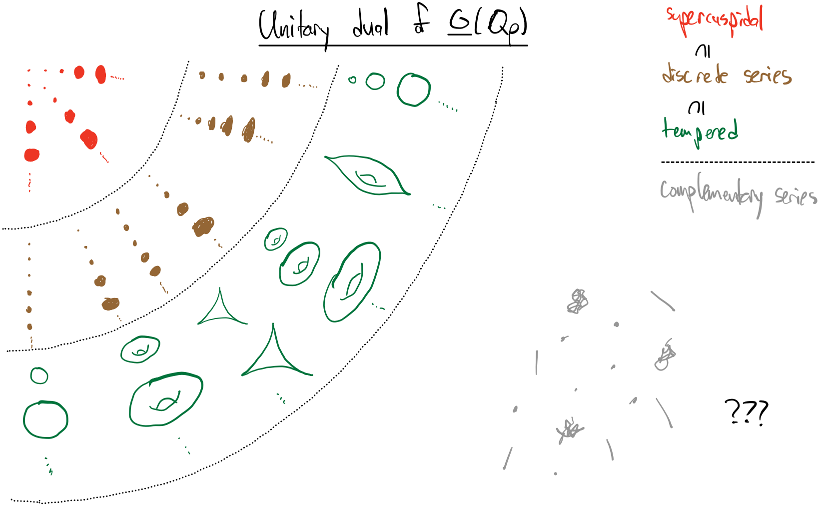
Example: hook-length formula

$$\dim V_{\yng(3,2,1)} = \frac{6!}{5 \cdot 3 \cdot 1 \cdot 3 \cdot 1 \cdot 1} = 16$$

5	3	1
3	1	
1		

A toy model for the formal degree conjecture!

Example: Reductive p-adic groups (G)



Formal degree

For unitary irreps,

- tempered: in support of Plancherel measure;
- discrete series: tempered and isolated in $\mathrm{Spec}_{\mathrm{u}}\, H$;
- formal degree (of a discrete series): its Plancherel measure.

Plancherel measure and formal degree depend on Haar measure.

Formal degree $\approx$ dimension:

- If $\dim \pi < \infty$, formal degree "equals" dimension.
- Even if $\dim \pi = \infty$, formal degree can be finite.

Part 2. Yu's supercuspidals

Moy-Prasad, Morris (1990's): depth-zero supercuspidals

Adler (1998): positive-depth (toral) supercuspidals

Yu (2001): supercuspidals for tame G

Kim (2007): Yu's construction is exhaustive if $p \gg 0$ (rel. G)

Fintzen (2018): Yu's construction is exhaustive if $p \nmid |W|$

- $p \leq n$ for classical of rank n (A_n, B_n, C_n, D_n)
- $p \leq 7$ for exceptional ($E_{6,7,8}$, F_4, G_2)

Summary: Yu constructed "almost all" supercuspidals

Yu's supercuspidals: heuristic definition

Ψ: input to Yu's construction (complicated!)

Heuristic (Kaletha): $\Psi \approx (S, \theta)$ where

- $\underline{S} \subseteq \underline{G}$ (tame) compact maximal torus
- $\theta\colon S \to \mathbb{C}^\times$ character

Key tools:

- Bruhat-Tits building (encodes subgroup structure)
- Moy-Prasad subgroups $G_{x,r}$ ($\approx$ congruence subgroups)

$$G = \mathbb{Q}_p^\times : \qquad \mathbb{Z}_p^\times \supset 1 + p\mathbb{Z}_p \supset 1 + p^2\mathbb{Z}_p \supset 1 + p^3\mathbb{Z}_p \supset \cdots$$

Formal degree of Yu's supercuspidals: formula

Ψ: input to Yu's constrution $\longmapsto$ supercuspidal π_Ψ

Theorem (S, 2020)

The formal degree of π_Ψ *is* $(\exp_p t \stackrel{\text{def}}{=} p^t)$

$$\frac{\dim \rho}{[G_x^0 : G_{x,0+}^0]} \exp_p \tfrac{1}{2}\left(\dim \underline{G} + \dim \underline{G}^0_{x,0:0+} + \sum_{i=0}^{d-1} r_i(|R_{i+1}| - |R_i|)\right).$$

All constants except r_i are uniformly bounded rel. G.

Within a "series" (fix S, vary $\theta\colon S \to \mathbb{C}^\times$), fdeg $\approx C \cdot p^{\text{depth}}$.

Formal degree of Yu's supercuspidals: proof sketch

$$\pi_\Psi = \text{c-Ind}_K^G \kappa \qquad\qquad \text{(compact induction)}$$

For H finite,

$$\dim \operatorname{Ind}_K^H \kappa = [H : K] \dim \kappa.$$

For G locally profinite, K compact-open,

$$\operatorname{fdeg} \text{c-Ind}_K^G \kappa = \frac{\dim \kappa}{\operatorname{vol} K}.$$

Compute $\operatorname{vol} K$ using Moy-Prasad theory.

Part 3. Formal degree conjecture

Conjecture (Langlands, 1970's; ...)

1. *There is a map*

$$\frac{\{\text{Smooth irreps of } G\}}{\text{iso.}} \twoheadrightarrow \frac{\{L\text{-parameters } \varphi\colon W' \to {}^L G\}}{\text{equiv.}}$$

   which is surjective with finite fibers (L-packets).

2. L-packet of $\varphi \simeq \operatorname{Spec_u} S_\varphi$ $\qquad (|S_\varphi| < \infty)$
3. *(Many more properties ...)*

$${}^L G \overset{\text{def}}{=} \widehat{G}(\mathbb{C}) \rtimes \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p), \qquad W' \approx \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$$

L-parameters $\supseteq$ Galois representations $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \operatorname{GL}(V)$

Formal degree conjecture: statement

Conjecture (Hiraga-Ichino-Ikeda, 2008)

π*: discrete series with parameter* (φ, ρ)*. Then*

$$\operatorname{fdeg} \pi = \frac{\dim \rho}{|S_\varphi|} \cdot |\gamma(\mathrm{Ad} \circ \varphi)|.$$

This is a "2-conjecture": it depends on the conjectural LLC.

$$W' \xrightarrow{\varphi} {}^L G \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\hat{\mathfrak{g}}), \qquad \gamma(\sigma) \stackrel{\mathrm{def}}{=} \varepsilon(0, \sigma) \cdot \frac{L(1, \sigma)}{L(0, \sigma)}$$

Formal degree conjecture: known cases

Full LLC:

- real Lie groups [Harish-Chandra]
- (inner forms of) GL_n [Silberger-Zink]
- (inner forms of) SL_n [Harris-Taylor, Henniart]
- classical groups [Beuzart-Plessis]

Partial LLC:

- unipotent discrete series [Lusztig, Reeder, Feng-Opdam-Solleveld]
- depth-zero supercuspidals [DeBacker-Reeder]

(not known to agree with Fargues-Scholze)

Kaletha's L-packets

Kaletha: epipelagic (2012) $\subset$ regular (2016) $\subset$ non-singular (2019)

If $p \nmid |W|$, "most" supercuspidals are non-singular (even regular).

Starting point: $(\underline{S}, \theta)$

On automorphic side, use Yu: $(\underline{S}, \theta) \approx \Psi \mapsto \pi_\Psi$

On Galois side, use functoriality:

$$\begin{tikzcd}
W \arrow[r, "{}^L\theta"] \arrow[dr, "\hat{\theta}"'] & {}^L S \arrow[r, dashed, "{}^L j_\chi"] & {}^L G \\
 & \widehat{S} \arrow[u, hook] \arrow[r, "\hat{\jmath}"] & \widehat{G} \arrow[u, hook]
\end{tikzcd}$$

Formal degree conjecture for regular supercuspidals

Theorem

Kaletha's regular (S, 2020) and non-singular (Ohara, 2021) L-packets satisfy the formal degree conjecture:

$$\operatorname{fdeg}\pi = \frac{\dim\rho}{|S_\varphi|} \cdot |\gamma(\mathrm{Ad}\circ\varphi)|.$$

Proof sketch:

- $\operatorname{fdeg}\pi$: use earlier formula
- $(\dim\rho)/|S_\varphi|$: harder in non-singular case
- $|\gamma(\mathrm{Ad}\circ\varphi)|$: decompose $\mathrm{Ad}\circ\varphi$, use additivity of γ

Key challenges are in depth zero!

Part 4. Gross and Reeder's refinement

What is the sign of γ in the formal degree conjecture?

Recall $\gamma(\sigma) \stackrel{\text{def}}{=} \varepsilon(0, \sigma) \cdot \dfrac{L(1, \sigma)}{L(0, \sigma)}$.

$$\varepsilon(s, \sigma) = \omega(\sigma) \cdot p^{(\operatorname{cond} \sigma)(1/2 - s)}$$

root number: mysterious, carries deep information

σ self-dual: $\omega(\sigma)^2 = (\det \sigma)(-1) = \pm 1$.

- σ symplectic: branching problems (GGP)
- σ orthogonal: central characters

Gross and Reeder's refinement

Starting point: normalize Haar measure so that

$$\mathrm{fdeg}(\mathrm{Steinberg}) = 1.$$

Conjecture (Gross-Reeder, 2010)

Say $Z(G)$ compact and π disc. ser. with L-parameter φ. Then

$$\frac{\omega(\varphi \circ \mathrm{Ad})}{\omega(\varphi_{\mathrm{prin}} \circ \mathrm{Ad})} = \pi(z_{\mathrm{Ad}}) \qquad (= \pm 1).$$

φ_{prin}: the L-parameter of Steinberg

$\mathrm{Ad} \mapsto z_{\mathrm{Ad}} \in Z(G)$ and $z_{\mathrm{Ad}}^2 = 1$

Orthogonal root numbers: formula

The conjecture is true mod expected properties of LLC:

Theorem (S, 2021)

Let $\varphi\colon W' \to {}^L G$ tempered, $r : {}^L G \to \mathrm{O}(V)$. Then

$$\frac{\omega(\varphi \circ r)}{\omega(\varphi_{\text{prin}} \circ r)} = \chi_\varphi(z_r) \qquad (= \pm 1).$$

$r \mapsto z_r \in Z(G)$ and $z_r^2 = 1$, $\qquad \chi_\varphi\colon Z(G) \to \mathbb{C}^\times$ (Langlands)

Conjecture: $\pi|_{Z(G)} = \chi_\varphi$

Strengths:

- No assumption that $Z(G)$ is compact.
- No assumption that "π" is disc. ser., only tempered.
- Holds uniformly for all local fields (incl. $\mathbb{R}$, $\mathbb{C}$, and char > 0).

Orthogonal root numbers: proof sketch

Goal: $\dfrac{\omega(\varphi \circ r)}{\omega(\varphi_{\mathrm{prin}} \circ r)} = \chi_\varphi(z_r) \qquad (= \pm 1).$

Key lemma: $r^*(c_{\mathrm{pin}}) = e_{r,*}(c_G) \cdot (p^* r|_W^*)(c_{\mathrm{pin}})$ in $\mathrm{H}^2({}^L G, \{\pm 1\})$.

Pull back along $\varphi\colon W \to {}^L G$; use $\mathrm{H}^2(W, \{\pm 1\}) \simeq \{\pm 1\}$.

- c_{pin} factors $\mapsto$ root numbers (Deligne's theorem)
- c_G factor $\mapsto \chi_\varphi(z_r)$

Technical challenge: local Tate duality for Weil group.

$$\mathrm{H}^2(W, X^*(\underline{T})) = \mathrm{Hom}(T^1, \mathbb{C}^\times) \qquad (T^1 = \text{max cpct subgp}).$$

Future work

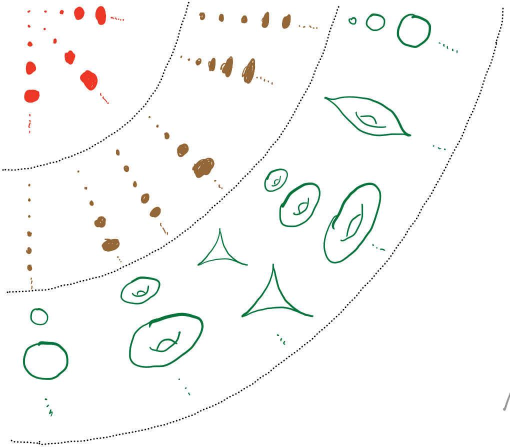
Folk conjecture: every supercuspidal is $\text{c-Ind}_K^G \kappa$ for some K, κ.

Same strategy for fdeg of other supercuspidals (Stevens, ...).

What is the Plancherel measure on the rest of the tempered dual?

(Does it satisfy the (generalized) formal degree conjecture?)

Unitary dual of $\underline{G}(\mathbb{Q}_p)$



supercuspidal
$\cap$
discrete series
$\cap$
tempered

complementary series

Thank you for your attention!

Abstract

The local Langlands correspondence is more than a correspondence: it promises an extensive dictionary between the representation theory of reductive p-adic groups and the arithmetic of their L-parameters. One entry in this dictionary is a conjectural formula of Hiraga, Ichino, and Ikeda for the size of a discrete series representation – its “formal degree” – in terms of a gamma factor of its L-parameter. In the first part of the talk, I'll explain why the conjecture is true for almost all supercuspidal representations. In the second part, I'll compute the sign of the gamma factor, verifying a conjecture of Gross and Reeder.