Fundamental Notions in Algebra – Exercise No. 5

1. Show that if $f : R \to S$ is a surjective ring homomorphism, then $f(J(R)) \subset J(S)$. Does this inclusion have to be an equality? What happens if f is not surjective?

2. Let R be a ring, and let M be an R-module.

   (a) Prove that M is semisimple if and only if every cyclic submodule of M is semisimple.

   (b) Prove that M is semisimple if and only if for all $m \in M$, $\mathrm{ann}(m)$ is an intersection of finitely many maximal ideals of R.

3. Let R be an artinian ring, and M be an R-module. Show that M is of finite length if and only if it is finitely generated.

4. Let R be a ring.

   (a) Show that $J(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(J(R))$.

   (b) Let $T_n(R) \subset \mathrm{Mat}_n(R)$ be the ring of upper-triangular matrices. Compute $J(T_n(R))$ (in terms of $J(R)$).

   (c) Compute $J(\mathbb{Z}/n\mathbb{Z})$.

   (d) For which R and n are the above rings semisimple?

5. (a) Let R be an artinian ring. Show that an ideal I of R is nilpotent if and only if it consists of nilpotent elements.

   (b) Show that the assertion of (a) remains true if one replaces "artinian" by "noetherian and commutative".

   (c) Give an example of a ring for which the assertion in (a) is false.

6. Let R be an artinian ring.

   (a) Show that R has only finitely many isomorphism classes of simple modules.

   (b) Does R have only finitely many maximal ideals?

   (c) Does the assertion in (b) hold if one assumes (in addition) that R is commutative?