Fundamental Notions in Algebra – Exercise No. 8

1. Let D be a finite-dimensional algebra over K. Show that
   (a) D is a division algebra if and only if the subalgebra $K[x] \subset D$ is a field for each $x \in D$.
   (b) D is a division algebra if and only if D is an integral domain, that is, does not have zero-divisors.
   (c) A finite-dimensional subalgebra of a division algebra is a division algebra.
   (d) A finite ring without zero divisors is a division ring.

2. Let A and B be commutative rings.

   Definition: A set P is called a (A, B)-bimodule, if P has a structure of an A-module and a B-module, which satisfy $(ax)b = a(xb)$ for all $a \in A, b \in B$ and $x \in P$.

   Let M be an A-module, N a B-module and P a (A, B)-bimodule. Show that $M \otimes_A P$ has a structure of a B-module, $P \otimes_B N$ has a structure of an A-module, and $(M \otimes_A P) \otimes_B N \cong M \otimes_A (P \otimes_B N)$.

3. Let A be a central simple algebra over K of dimesion n^2. Show that
   (a) $\operatorname{ind}(A) \mid n$, and $\operatorname{ind}(A) = n$ is and only if D is a division algebra.
   (b) $\operatorname{ind}(A) = \min\{[L : K] \mid L \text{ splits } A\} = \gcd\{[L : K] \mid L \text{ splits } A\}$, where gcd means "the greatest common divisor".
   (c) For every finite extension L/K, we have $\operatorname{ind}(A_L) \mid \operatorname{ind}(A)$ and $\operatorname{ind}(A) \mid [L : K] \operatorname{ind}(A_L)$.

4. Let D_1 and D_2 be finite dimensional division algebras over a field K (not neccesary central) such that $(\dim_K D_1, \dim_K D_2) = 1$. Show that $D_1 \otimes_K D_2$ is a division algebra in the following way:
   (a) Show first the assertion when D_1 and D_2 are fields.
   (b) Show the assertion when D_1 is central over K and D_2 is a field. (Use exercise 3).
   (c) Set $L_1 := Z(D_1)$ and $L_2 := Z(D_2)$. Deduce the general case using isomorphism $D_1 \otimes_K D_2 \cong D_1 \otimes_{L_1} L_1 \otimes_K L_2 \otimes_{L_2} D_2$.

5. Let L/K be a finite Galois extension of degree n, and set $G = Gal(L/K)$. The group G acts on L and we denote by $L\langle G\rangle$ the non-commutative "twisted" group ring consisting of the elements $\left\{\sum_{g \in G} b_g g \mid b_g \in L\right\}$ with the product given by the rule $gb = g(b)g$ (or equivalently, $(b_g g)(b_h h) = b_g g(b_h) gh$).

   Show that $L\langle G\rangle$ is isomorphic to $\mathrm{Mat}_n(K)$.