Seminar S4D2 SS 2024
Graduate Seminar Topology Stefan Schwede
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Equivariant homotopy theory

Tuesdays, 14:15-15:45 in SR 0.003

In this seminar we study equivariant homotopy theory, ultimately aiming at *stable* equivariant phenomena. The focus will be on actions by finite groups. General references are [Ad, Sch1, tD2]. Prerequisites are good knowledge of non-equivariant homotopy theory; previous knowledge of non-equivariant *stable* homotopy theory is helpful, but not required. To simplify our life we restrict our attention to finite groups (as opposed to the more general setting of compact Lie groups).

09.04.24 **G-spaces, G-equivalences and the equivariant Whitehead theorem** A. Flores
This talk develops the basic equivariant homotopy theory of G-spaces, for any (discrete) group G. In particular, definitions and basic properties of G-spaces and G-CW-complexes should be covered. Both G-weak equivalences and G-homotopy equivalences should be explained. The main result of the talk is the equivariant Whitehead theorem, saying that G-weak equivalences between G-CW-complexes are G-homotopy equivalences [tD2, Ch. II, Prop. 2.7] [Lü, Ch. I, Thm. 2.4].

16.04.24 **Reconstruction from fixed point data: Elmendorf's theorem** K. Chen
The equivariant Whitehead theorem can be interpreted as saying that the collection of fixed points for all subgroups of G determine the equivariant homotopy type. Elmendorf's theorem [El] makes a more precise statement how G-equivariant homotopy theory is equivalent to the homotopy theory of 'fixed point diagrams'. This talk should explain and prove this theorem, and illustrate it by giving examples (universal spaces for families of subgroups, Eilenberg-MacLane spaces for coefficient systems).

23.04.24 **Equivariant orthogonal spectra** T. Chatzidiamantis
There are different ways to set up a good theory of 'stable' equivariant phenomena; we will use the concept of *orthogonal G-spectra*, discussed for example in [MM, Sch1] and [Sch2, Sec. 3.1]. Besides the definition, this talk should explain equivariant stable homotopy groups of orthogonal G-spectra, equivariant stable equivalences, and present examples.

30.04.24 **Equivariant homology theories** C. Foth
This talk should discuss the suspension isomorphism (by a G-representation) for equivariant homotopy groups and the long exact mapping cone sequence arising from a morphism of orthogonal G-spectra (see for example [Sch1, Sec. 3] or [Sch2, Sec. 3.1]). Based on this, a proof should be given that every orthogonal G-spectrum defines an equivariant homology theory on the category of G-spaces. An explanation in which way these homology theories are '$RO(G)$-graded' would be desirable.

07.05.24 **The Wirthmüller isomorphism** P. Srivastava
Let H be a subgroup of a finite group G, and Y an orthogonal H-spectrum. The Wirthmüller isomorphism [Wi] identifies the G-equivariant homotopy groups of the induced G-spectrum $G_+ \wedge_H Y$ with the H-equivariant homotopy groups of Y. Another reference is [Sch1, Sec. 4].

14.05.24 **Mackey functors: Restriction and transfer in equivariant homotopy groups**
E. Cortinovis
This talk should introduce the algebraic concept of a G-*Mackey functor*, ideally from two different perspectives (i.e., via 'generators and relations' and as additive functors on spans of finite G-sets). Then restriction and transfer maps should be defined for the equivariant homotopy groups of orthogonal G-spectra; time permitting, a proof of the Mackey functor properties should be sketched.

28.05.24 **The tom Dieck splitting** C. Liu
The tom Dieck splitting [tD1, Satz 4] expresses the equivariant homotopy groups of a suspension spectrum of a G-space X as a direct sum, indexed by conjugacy classes of subgroups H, of contributions coming from the H-fixed points X^H. State and prove the tom Dieck splitting; alternative references are [tD2, II Thm. 7.7] or [Sch1, Sec. 6].

04.06.24 **The equivariant zero stem** B. Mandal
Establish the isomorphism between the Burnside ring of a finite group G and the 0-th equivariant stable stem $\pi_0^G(\mathbb{S})$, given by the transfer maps in equivariant stable homotopy groups, originally due to Segal [Se]. Show that the isomorphism is compatible with restriction and transfers. Describe explicit representatives for the transfer classes $\mathrm{tr}_H^G(1)$. Alternative references are [tD2, II Sec. 8] or [Sch1, Thm. 6.16]. Define the degree homomorphism into the ring of integral class functions and prove the congruences that characterize the image.

11.06.24 **Geometric fixed points** J. Zhang
Define the geometric fixed point spectra and the geometric fixed point map. Identify geometric fixed point homotopy groups with equivariant homotopy groups of the smash product with the universal space for proper subgroups. Show that geometric fixed point homotopy groups cocincide with the localization of genuine equivariant homotopy groups at the Euler classes of orthogonal representations. Possible references are [Sch1, Sec. 7] and [Sch2, Sec. 3.3].

18.06.24 **Equivariant Thom spectra** Y. Song
There are different equivariant forms of the Thom spectrum whose homotopy groups are the unoriented bordism groups. Discuss the G-equivariant orthogonal spectra $\mathbf{mO}$ and $\mathbf{MO}$ [Sch2, Sec. 6.1]. Calculate the geometric fixed points of $\mathbf{mO}$ and $\mathbf{MO}$ in terms of non-equivariant bordism groups of Grassmannians, compare [Sch2, Ex. 6.1.46]. Show that the equivariant homology theory represented by $\mathbf{MO}$ is a localization of that represented by $\mathbf{mO}$, obtained by inverting the 'inverse Thom classes', see [Sch2, Cor. 6.1.35].

25.06.24 **Equivariant bordism** F. Neugebauer
Define the bordism groups of smooth closed G-manifolds, and explain how and why they form an equivariant homology theory; one possible reference is [Sch2, Sec. 6.2]. Define the equivariant Thom-Pontryagin map from the G-bordism theory to the G-homology theory represented by $\mathbf{mO}$, and show that it is an isomorphism for all finite groups G, using tom Dieck's proof in [tD1, Satz 5], or specializing the proof in [Sch2, Thm. 6.2.33] to finite groups.

02.07.24 **Higher categorical description of genuine G-spectra** M. Lorenzi
Explain the higher categorical description of the ∞-category of genuine G-spectra as the localization of the ∞-category of G-spaces by inverting representation spheres, following

[GM, Appendix C]. At least the ‘additive’ version in [GM, Proposition C.4] should be covered, ideally also the ’multiplicative’ version in [GM, Proposition C.9]. This talk differs form the other ones in that basic knowledge about higher category theory is essential.

References

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[El] A D Elmendorf, *Systems of fixed point sets.* Trans. Amer. Math. Soc. 277 (1983), no. 1, 275–284.

[GM] D Gepner, L Meier, *On equivariant topological modular forms.* Compos. Math. 159 (2023), no. 12, 2638–2693.

[Lü] W Lück, *Transformation groups and algebraic K-theory.* Lecture Notes in Math., 1408, Springer, 1989

[MM] M A Mandell, J P May, *Equivariant orthogonal spectra and S-modules.* Mem. Amer. Math. Soc. 159 (2002), no. 755, x+108 pp.

[Sch1] S Schwede, *Lecture notes on equivariant stable homotopy theory.* `http://www.math.uni-bonn.de/people/schwede/equivariant.pdf`

[Sch2] S Schwede, *Global homotopy theory.* New Mathematical Monographs 34. Cambridge University Press, 2018. xviii+828 pp.

[Se] G B Segal, *Equivariant stable homotopy theory.* Actes du Congrès International des Mathematiciens. (Nice, 1970), Tome 2, pp. 59–63.

[tD1] T tom Dieck, *Orbittypen und äquivariante Homologie. II.* Arch. Math. (Basel) 26 (1975), no. 6, 650–662.

[tD2] T tom Dieck, *Transformation groups.* de Gruyter Studies in Mathematics **8**, de Gruyter, 1987

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