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$T(1)$ and $T(b)$ theorems and applications

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1 Tb theorem on non-homogeneous spaces I

after F. Nazarov, S. Treil and A. Volberg [1]
A summary written by Polona Durcik

Abstract

We discuss a Tb-theorem which extends the Tb-theorem by David, Journé and Semmes for the Calderón-Zygmund operators on $\mathbb{R}^n$ to the case of non-doubling measures.

1.1 Introduction

Let μ be a Borel measure on $\mathbb{R}^n$ and d a positive number. The measure μ may be non-doubling, we assume only that $\mu(B(x,r)) \leq r^d$ for any ball $B(x,r)$ with radius r and center x. A *Calderón-Zygmund kernel (of dimension d)* is a function $K \in L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y) : x = y\}, \mu)$ satisfying

(i) $|K(s,t)| \leq C|s-t|^{-d}$

(ii) There exists $\alpha > 0$ and $C > 0$ such that whenever $|t - s_0| \geq 2|s - s_0|$,

$$|K(s,t) - K(s_0,t)|, |K(t,s) - K(t,s_0)| \leq C\frac{|s-s_0|^\alpha}{|t-s_0|^{d+\alpha}}$$

We are interested in the $L^p(\mu)$ boundedness of a *Calderón-Zygmund operator* T (integral operator with kernel K). Being an integral operator with kernel K means that the bilinear form $\langle Tf, g\rangle$ of T (or $\langle Tb_1 f, b_2 g\rangle$ when talking about $b_2 T b_1$) is well defined for some class of functions (say, C_0^∞ functions) and that for compactly supported f, g with disjoint supports

$$\langle Tf, g\rangle = \int\int K(x,y)g(x)f(y)d\mu(x)d\mu(y) \tag{1}$$

We call a bounded (complex valued) function b *weakly accretive* (with respect to μ) if there exists a $\delta > 0$ such that for any cube[1] Q

$$\mu(Q)^{-1}\left|\int_Q b(s)d\mu(s)\right| \geq \delta$$

This in particular implies $|b| \geq \delta$ $\mu-$a.e. The main result of [1] is

[1]By a cube we mean an object obtained from $[0,1)^n$ by dilations and shifts.

Theorem 1 ($Tb-$theorem)**.** *Let $1 < p < \infty$ and let b_1, b_2 be two weakly accretive functions. A Calderón-Zygmund operator T is bounded on $L^p(\mu)$ if and only if the operator $b_2 T b_1$ is weakly bounded and Tb_1, T^*b_2 belong to $BMO(\mu)$.*

Moreover, the upper bound of the norm of T depends on n, d, p CZ constants of the kernel K, $\|b\|_\infty$, δ and the BMO norm of Tb.

This theorem should be seen as a *meta-theorem*. There are several interpretations of weak boundedness which depend on the initial assumptions on T. According to that we will state different (preciser) versions of the theorem again. When we assume that T is well defined for compactly supported functions, one should think that instead of T we are given a family $(T_\varepsilon)_\varepsilon$ of truncated operators $T_\varepsilon f(x) := \int_{|x-y|>\varepsilon} K(x,y)f(y)d\mu(y)$ and think of boundedness of T as uniform boundedness of T_ε.

As in the homogeneous case it suffices to show boundedness on $L^2(\mu)$, for this cf. [2] where weak $1-1$ estimates for Calderón-Zygmund operators on non-homogeneous spaces are proven.

1.2 BMO spaces

We also have to say what is the BMO space in the theorem. There are many spaces which generalize the case when μ is a $n-$dimensional Lebesgue measure in $\mathbb{R}^n$ and all of the definitions below then give the well known classical BMO.

1.2.1 BMO_λ^p

Let $1 \le p < \infty$ and $\lambda > 1$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ belongs to $\mathrm{BMO}_\lambda^p(\mu)$ if there exists a constant C such that

$$\Big(\int_Q |f - m_Q(f)|^p d\mu\Big)^{1/p} \le C\mu(\lambda Q)^{1/p}$$

for all cubes Q, where $m_Q f = \mu(Q)^{-1}\int_Q f d\mu$ is the average of f over Q. The best constant C is defined to be $\|f\|_{\mathrm{BMO}_\lambda^p(\mu)}$. Here λQ means the cube Q dilated λ times with respect to its center. In order to find the best generalization of the classical BMO, $\mathrm{BMO}_\lambda^p(\mu)$ has some disadvantages. It depends on λ and p and one can show that the inclusions $\mathrm{BMO}_\lambda^p(\mu) \subset \mathrm{BMO}_\Lambda^p(\mu)$ if $\lambda < \Lambda$ and $\mathrm{BMO}_\lambda^{p_2}(\mu) \subset \mathrm{BMO}_\lambda^{p_1}(\mu)$ if $p_1 < p_2$ are proper. Also, $\mathrm{BMO}_1^p(\mu)$

is a wrong object for this theory, since boundedness of T on $L^p(\mu)$ does not imply $T1 \in \mathrm{BMO}_1^{\mathrm{p}}(\mu)$. The space RBMO of X. Tolsa turns out to be a more natural analogue.

1.2.2 RBMO

Let $\rho > 1$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is in $\mathrm{RBMO}(\mu)$ (regularized BMO) if for each cube Q there exists a number f_Q such that

$$\int_Q |f - f_Q| \leq B_1 \mu(\rho Q)$$

and such that for all cubes $Q,\ R$ with $Q \subset R$

$$|f_R - f_Q| \leq B_2 \Big(1 + \int_{2R \setminus Q} \frac{d\mu(x)}{|x - c_Q|^d} \Big)$$

where the constants $B_1,\ B_2$ do not depend on Q. The infimum of $B_1 + B_2$ is called the RBMO$-$norm of f.

The second regularization condition is crucial for better behaviour of RBMO over BMO_λ^p. It may seem that this space depends on the parameter ρ, but one can show that it does not. And most importantly, RBMO has the John-Nirenberg property (unlike BMO_λ^p):

Theorem 2. *Let $f \in \mathrm{RBMO}$, $\rho > 1$ and $1 \leq p < \infty$. Then for any cube Q*

$$\int_Q |f - f_Q|^p d\mu \leq B(\rho, p, \|f\|_{\mathrm{RBMO}}) \mu(\rho Q)$$

1.2.3 $Tb \in \mathrm{BMO}$

Next we need to make sense of what it means that Tb belongs to a BMO space for $b \in L^\infty$.

Let us suppose that the bilinear form $\langle Tb_1 f, b_2 g\rangle$ of the operator $b_2 T b_1$ is defined for $f, g \in C_0^\infty$. Let φ be an arbitrary smooth function supported on a cube Q satisfying $\int b_2 \varphi = 0$. Let $\psi_1 \in C_0^\infty$, $\psi_1 \equiv 1$ on $2Q$ with $0 \leq \psi \leq 1$. Let $\psi_2 = 1 - \psi_1$. We define the expression $\langle Tb_1, \varphi b_2\rangle$ to be $\langle T\psi_1 b_1, \varphi b_2\rangle + \langle T\psi_2 b_1, \varphi b_2\rangle$. While the first term is defined by assumption, for the second term we use an integral representation by analogy with (1).

This representation does not depend on the choice of ψ_1 and we can interpret the condition $Tb_1 \in \mathrm{BMO}_\lambda^p$ by duality as

$$|\langle Tb_1, \varphi b_2\rangle| \leq C\|\varphi b_2\|_{L^{p'}(\mu)}\mu(\lambda Q)^{1/p}$$

where $1/p + 1/p' = 1$.

The condition $Tb_1 \in$ RBMO needs a different interpretation. Let us suppose that the operator T is well defined on bounded compactly supported functions. We say that f belongs to RBMO(G,μ) if the inequalities defining RBMO hold for all cubes $Q \subset R \subset G$. We consider $\varphi \in C_0^\infty$, $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on the cube $10G$. We say that $Tb_1 \in \mathrm{RBMO}(G, \mu)$ if $Tb_1\varphi \in \mathrm{RBMO}(G, \mu)$, which is independent of a cutoff φ. Finally, we say that $Tb_1 \in \mathrm{RBMO}(\mu)$ if $Tb_1 \in \mathrm{RBMO}(G, \mu)$ for all cubes Q with uniform estimates on the norms.

Since RBMO has the John-Nirenberg property, if $Tb_1 \in \mathrm{RBMO}(\mu)$ then $Tb_1 \in \mathrm{BMO}_\lambda^p(\mu)$ for all $p \in [1, \infty)$, $\lambda > 1$.

The condition $Tb_1 \in$ RBMO may be sometimes hard to verify. But for our theorem it does not matter, which BMO space we pick. If $Tb_1 \in \mathrm{BMO}_\lambda^p(\mu)$ for some p and b_2Tb_1 is weakly bounded in the sense that there exists $\Lambda > 1$ such that $|\langle Tb_1\chi_Q, b_2\chi_Q\rangle| \leq C\mu(\Lambda Q)$ for all cubes Q, then $Tb_1 \in \mathrm{RBMO}(\mu)$.

1.3 Estimates of the regular part of the matrix

Fix two dyadic lattices $\mathcal{D}, \mathcal{D}'$ in $\mathbb{R}^n$ consisting of cubes of size 2^k, $k \in \mathbb{Z}$, where one is shifted with respect to the other. One version of Theorem 1 (the "if" part) with a stronger weak boundedness assumption is the following:

Theorem 3. *Let T be a Calderón-Zygmund operator which is bounded on compactly supported functions, i.e. for compactly supported f, g*

$$|\langle Tf, g\rangle| \leq C(A)\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}$$

where $A = \max\{\mathrm{diam}(\mathrm{supp}\, f), \mathrm{diam}(\mathrm{supp}\, g)\}$. Let b_1, b_2 be weakly accretive functions and let $Tb_1, Tb_2 \in \mathrm{BMO}_\lambda^2(\mu)$. Suppose also that T is weakly bounded in the sense that

$$|\langle Tb_1\chi_Q, b_2\chi_R\rangle| \leq C\mu(Q)^{1/2}\mu(R)^{1/2}$$

for cubes $Q \in \mathcal{D}, R \in \mathcal{D}'$ of comparable size which are close, i.e. Q, R such that $1/2 \leq \ell(Q)/\ell(R) \leq 2$ and $\mathrm{dist}(Q, R) \leq \min(\ell(Q), \ell(R))$.

Then the operator T is bounded on $L^2(\mu)$ with the upper bound of the norm of T depending only on the constants as in Theorem 1.

Let us sketch the proof. We want to estimate $|\langle Tf, g\rangle| \leq C\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}$ for $f, g \in L^2(\mu)$. First we decompose f and g into martingale differences: For a weakly accretive function b we define

$$E^b_Q f(x) := \Big(\int_Q b\,d\mu\Big)^{-1}\Big(\int_Q f d\mu\Big)\cdot b(x)\cdot \chi_Q(x)$$

and $\Delta^b_Q := E^b_{Q'} - E^b_Q$, where $Q' \subset Q$ and $\ell(Q') = 1/2\ell(Q)$. Then for a fixed $k \in \mathbb{Z}$, any $f \in L^2(\mu)$ can be decomposed as

$$f = \sum_{Q\in\mathcal{D},\ell(Q)\leq 2^k} \Delta^b_Q f + \sum_{Q\in\mathcal{D},\ell(Q)=2^k} E^b_Q f \tag{2}$$

where the series converges in $L^2(\mu)$.

Let us call a pair of cubes Q, R with $\ell(Q) \leq \ell(R)$ *singular* if $\operatorname{dist}(Q, \partial R) \leq \ell(Q)^\gamma \ell(R)^{1-\gamma}$ or $\operatorname{dist}(Q, \partial R_k) \leq \ell(Q)^\gamma \ell(R_k)^{1-\gamma}$ for some subcube $R_k \subset R$ of size $\ell(R)/2$, where γ is such that $\gamma d + \gamma\alpha = \alpha/2$. We call a singular pair Q, R *essentially singular*, if in addition $\ell(Q) < 2^{-r}\ell(R)$ for some integer r to be determined later. We say that a cube $Q \in \mathcal{D}$ is *bad* if there exists a bigger cube $R \in \mathcal{D}'$ such that the pair Q, R is essentially singular. Otherwise it is called *good*. Let $f \in L^2(\mu)$ be supported by a cube of size 2^k. We call the function f $\mathcal{D}-$*good* if $\Delta^{b_1}_Q f = 0$ for any bad cube $Q \in \mathcal{D}$ of size $\ell(Q) < 2^k$. If we replace $\mathcal{D}$ by $\mathcal{D}'$ and b_1 with b_2 we get a definition for $\mathcal{D}'-$*good* functions.

Fix r from the definition of singular pairs large enough such that $2^{r(1-\gamma)} \geq \lambda$ and $2^r > 4\lambda$. After decomposing f and g into martingale differences we first estimate $\sum_{Q,R}\langle T\varphi_Q, \psi_R\rangle$ where $\varphi_Q = \Delta^{b_1}_Q f$ and $\psi_R = \Delta^{b_2}_R g$. For now we only treat the case when f is $\mathcal{D}-$good and g is $\mathcal{D}'-$good, so all entries in $\langle T\varphi_Q, \psi_R\rangle$ corresponding to essentially singular pairs are zero. Then:

Case I: Q and R are "far away" from each other

By this we mean that $\operatorname{dist}(Q, R) \geq \min(\ell(Q), \ell(R))$ and $2^{-r}\ell(R) \leq \ell(Q) \leq 2^r\ell(R)$, or that Q and R are disjoint, nonsingular and that (by symmetry) $\ell(Q) < 2^{-r}\ell(R)$. In this case we can use (1) and estimate

$$|\langle T\varphi_Q, \psi_R\rangle| \leq C\frac{\ell(Q)^{\alpha/2}\ell(R)^{\alpha/2}}{D(Q,R)^{\alpha+d}}\mu(Q)^{1/2}\mu(R)^{1/2}\|\varphi_Q\|_{L^2(\mu)}\|\psi_Q\|_{L^2(\mu)}$$

where $D(Q, R) := \operatorname{dist}(Q, R) + \ell(Q) + \ell(R)$. Then we show that the matrix $\{T_{Q,R}\}_{Q\in\mathcal{D},Q'\in\mathcal{D}'}$ defined via

$$T_{Q,R} := \frac{\ell(Q)^{\alpha/2}\ell(R)^{\alpha/2}}{D(Q,R)^{\alpha+d}}\mu(Q)^{1/2}\mu(R)^{1/2}$$

generates a bounded operator on ℓ^2 and we are done.

Case II: $Q \subset R$ and Q is not close to ∂R

We consider $Q \subset R$ with $\ell(Q) < 2^{-r}\ell(R)$ where the pair is not singular. First we define a paraproduct $\Pi = \Pi_{T^*}$ by

$$\Pi f := \sum_{R\in\mathcal{D}'} \sum_{\substack{Q\in\mathcal{D}:\ell(Q)=2^{-r}\ell(R) \\ \mathrm{dist}(Q,\partial R)\geq\lambda\ell(Q)}} (E_R b_2)^{-1} \cdot E_R f \cdot (\Delta_Q^{b_1})^* T^* b_2$$

Then we decompose

$$\langle T\varphi_Q, \psi_R\rangle = \langle (T-\Pi^*)\varphi_Q, \psi_R\rangle + \langle \varphi_Q, \Pi\psi_R\rangle$$

If $T^*b_2 \in \mathrm{BMO}^2_\lambda(\mu)$, Π is bounded on $L^2(\mu)$. This can be shown using a dyadic version of the Carleson embedding theorem. To estimate $\langle (T-\Pi^*)\varphi_Q, \psi_R\rangle$ note that the function ψ_R is of the form

$$\psi_R(x) = \sum_{i=1}^{2^n} B_i \cdot \chi_{R_i}(x) \cdot b_2(x)$$

where B_i are some constants and $R_i \in \mathcal{D}'$ are the dyadic cubes of size $\ell(R)/2$ contained in R. We also have $\langle \varphi_Q, \Pi\psi_R\rangle = \langle T\varphi_Q, b_2\rangle B_1$. This makes it possible to estimate $\langle (T-\Pi^*)\varphi_Q, \psi_R\rangle$ in a similar way as in case I.

Case III: Q and R are close and of comparable size

This is the case when $\mathrm{dist}(Q,R) \leq \min(\ell(Q),\ell(R))$ but Q and R are still of comparable size, i.e. $2^{-r}\ell(R) \leq \ell(Q) \leq 2^r\ell(R)$. So the pair Q,R is singular but not essentially singular. The claim can be now deduced by the weak boundedness assumption.

Terms involving $E_Q^{b_i} f$ are treated similarly. What remains is to prove the theorem for bad functions f and g, which is the harder part of this proof.

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2 Painlevé's problem and the semiadditivity of analytic capacity

after Xavier Tolsa [8]
A summary written by Daniel Girela-Sarrión

Abstract

We prove that, for $E \subset \mathbb{C}$ a compact set, $\gamma(E) \approx \gamma_+(E)$, solving Painlevé's problem. This result also implies the semiadditivity of analytic capacity.

2.1 Introduction

Definition 1. *A compact set $E \subset \mathbb{C}$ is said to be removable for bounded analytic functions (or, simply, removable) if, whenever Ω is an open set containing E, every bounded analytic function in $\Omega \setminus E$ has an analytic extension to the whole of Ω.*

Painlevé [7] proved in 1888 that if a set has zero 1-dimensional Hausdorff measure, then it is removable. The classical **Painlevé problem** consists in giving a geometric/metric characterization of removable sets.

To study this problem, Ahlfors introduced the notion of analytic capacity.

Definition 2. *Let $E \subset \mathbb{C}$ be a compact set. The analytic capacity of E, denoted by $\gamma(E)$, is defined by*

$$\gamma(E) = \sup\{|f'(\infty)| \colon f \in \mathcal{H}ol(\mathbb{C} \setminus E),\ |f(z)| \leq 1 \text{ for all } z \in \mathbb{C} \setminus E\}.$$

Ahlfors [1] proved that a compact set $E \subset \mathbb{C}$ is removable for bounded analytic functions if, and only if, $\gamma(E) = 0$.

Later, in the 1960s, the notion of analytic capacity was rediscovered by Vitushkin [11], who used it for problems of rational approximation in compact sets. Because of the applications to this type of problems, Vitushkin raised the question of the **semiadditivity** of γ, i.e., whether there exists an absolute constant c such that

$$\gamma(E \cup F) \leq c(\gamma(E) + \gamma(F))$$

for all compact sets $E, F \subset \mathbb{C}$.

2.2 Cauchy transforms and the capacity γ_+.

Definition 3. *If ν is a complex measure in $\mathbb{C}$, the Cauchy transform of ν is the function defined by*

$$C\nu(z) = \int \frac{1}{\xi - z} d\nu(\xi),$$

whenever the integral makes sense.

Cauchy transforms can be considered as a tool for constructing analytic functions. Indeed, in the conditions of the definition,

- $C\nu \in L^1_{loc}(\mathbb{C})$.
- $C\nu$ is analytic in $\mathbb{C} \setminus \mathrm{supp}(\nu)$.
- $C\nu(\infty) = 0$, $(C\nu)'(\infty) = -\nu(\mathbb{C})$.

Definition 4. *Given a compact set $E \subset \mathbb{C}$, we define the capacity γ_+ of E by*

$$\gamma_+(E) = \sup\{\mu(E) \colon \mathrm{supp}(\mu) \subset E, ||C\mu||_{L^\infty}(\mathbb{C}) \leq 1\}.$$

This capacity was introduced by Murai [5], only for sets supported in rectifiable curves, and he showed its relationship with the weak (1,1)-boundedness of the Cauchy transform on these curves. It is immediate, from the previous remarks, that $\gamma_+(E) \leq \gamma(E)$.

The usefulness of this capacity γ_+ stems from the fact that it can be characterized in terms of L^2-boundedness of Cauchy transforms, curvature of measures or certain potentials. Let us introduce some notation to state this characterizations.

If ν is a complex measure in $\mathbb{C}$, the integral

$$\int \frac{1}{\xi - z} d\nu(\xi)$$

may not be convergent for $z \in \mathrm{supp}(\nu)$. For this reason, one considers the truncated Cauchy transform of ν, which is defined by

$$C_\epsilon\nu(z) = \int_{|z-\xi|>\epsilon} \frac{1}{\xi - z} d\nu(\xi)$$

for all $z \in \mathbb{C}$ and all $\epsilon > 0$. If μ is a fixed positive Radon measure in $\mathbb{C}$, we write $C_\mu f = C(f\mu)$ and $C_{\mu,\epsilon} f = C_\epsilon(f\mu)$. We say that the Cauchy transform is bounded in $L^2(\mu)$ if all the $C_{\mu,\epsilon}$'s are bounded in $L^2(\mu)$ uniformly on $\epsilon > 0$, and we set

$$||C_\mu||_{L^2(\mu)\to L^2(\mu)} = \sup_{\epsilon>0} ||C_{\mu,\epsilon}||_{L^2(\mu)\to L^2(\mu)}.$$

A positive Radon measure μ is said to have linear growth if there exists a constant C such that

$$\mu[B(x,r)] \leq Cr$$

for all $x \in \mathbb{C}$ and all $r > 0$.

Given three pairwise different points $x, y, z \in \mathbb{C}$, its Menger curvature is defined by

$$c(x,y,z) = \frac{1}{R(x,y,z)},$$

where $R(x,y,z)$ is the radius of the circle passing through x, y, z. If μ is a positive Radon measure, we set

$$c_\mu^2(x) = \iint c(x,y,z)^2 d\mu(y) d\mu(z)$$

and we define the total curvature of μ by

$$c^2(\mu) = \int c_\mu^2(x) d\mu(x) = \iiint c(x,y,z)^2 d\mu(x) d\mu(y) d\mu(z).$$

Finally, the maximal radial Hardy-Littlewood operator is defined by

$$M\nu(z) = \sup_{r>0} \frac{|\nu|[B(z,r)]}{r}, \quad z \in \mathbb{C},$$

where ν is a complex measure in $\mathbb{C}$.

With all this notation, we can characterize the capacity γ_+ in many ways, as the following theorem states.

Theorem 5. *Let $E \subset \mathbb{C}$ be a compact set. We denote by $\Sigma(E)$ the set of all positive Radon measures μ supported on E and such that $\mu[B(x,r)] \leq r$ for all $x \in \mathbb{C}$ and $r > 0$. Also, if μ is a positive Radon measure supported on E, we define*

$$U_\mu(x) = M\mu(x) + c_\mu^2(x)^{\frac{1}{2}}.$$

Finally, we set

$$M_+(\mathbb{C}) = \{\mu\colon \mu \textit{ is a positive Radon measure in } \mathbb{C}\}.$$

Then, we have

$$\begin{aligned}
\gamma_+(E) &\approx \sup\{\mu(E)\colon \mu \in \Sigma(E), \|C_\epsilon \mu\|_{L^\infty(\mu)} \le 1 \textit{ for all } \epsilon > 0\} \\
&\approx \sup\{\mu(E)\colon \mu \in \Sigma(E), \|C_\epsilon \mu\|^2_{L^2(\mu)} \le \mu(E) \textit{ for all } \epsilon > 0\} \\
&\approx \sup\{\mu(E)\colon \mu \in \Sigma(E), c^2(\mu) \le \mu(E)\} \\
&\approx \sup\{\mu(E)\colon \|C_\mu\|_{L^2(\mu)\to L^2(\mu)} \le 1\} \\
&\approx \sup\{\mu(E)\colon \operatorname{supp}(\mu) \subset E, U_\mu(x) \le 1 \textit{ for all } x \in E\} \\
&\approx \sup\{\mu(E)\colon \operatorname{supp}(\mu) \subset E, U_\mu(x) \le 1 \textit{ for all } x \in \mathbb{C}\} \\
&\approx \inf\{\|\mu\|\colon \mu \in M_+(\mathbb{C}), U_\mu(x) \le 1 \textit{ for all } x \in E\}.
\end{aligned}$$

The semiadditivity of the capacity γ_+ follows from this result.

Corollary 6. *There exists an absolute constant $C > 0$ such that, for all compact sets $E, F \subset \mathbb{C}$,*

$$\gamma_+(E \cup F) \le C(\gamma_+(E) + \gamma_+(F)).$$

2.3 The comparability between γ and γ_+ and the semiadditivity of analytic capacity.

Since γ_+ is semiadditive, the semiadditivity of γ would follow from the comparability between γ and γ_+. Also, since the compact sets E with $\gamma_+(E) = 0$ are easily characterized in a metric/geometric way (indeed, from Theorem 5, $\gamma_+(E) = 0$ if, and only if, E supports a non-zero positive Radon measure μ with linear growth and finite curvature), this would also lead to a complete solution of Painlevé problem.

The main tools to attack this problem have been the local $T(b)$-type theorems for the Cauchy transform, originally due to Christ [2] in the setting of homogeneous spaces, and its refinement in the non-homogeneous case due to Nazarov, Treil and Volberg [6].

The first result in this setting is due to David [3].

Theorem 7. *Let $E \subset \mathbb{C}$ be a compact set with finite length and $\gamma(E) > 0$. Then, $\gamma_+(E) > 0$.*

Later, Mateu, Verdera and Tolsa [4] obtained precise estimates for the analytic capacity of a big class of planar Cantor sets, proving that γ and γ_+ are comparable for these sets.

Finally, Tolsa [8] proved, in 2001, the following:

Theorem 8. *There exists an absolute constant $c > 0$ such that for all compact sets $E \subset \mathbb{C}$,*

$$\gamma(E) \leq c\gamma_+(E).$$

As we have stated before, this result yields to the solution of Painlevé's problem and to the semiadditivity of analytic capacity.

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3 The solution of the Kato square root problem for the second order elliptic operators on $\mathbb{R}^n$

after P. Auscher, S. Hofmann, M. Lacey, A. M^{c}Intosh and P. Tchamitchian [1]

A summary written by Ana Grau de la Herrán

Abstract

The Kato problem questions for which operators the estimate $\|\sqrt{L}f\|_2 \sim \|\nabla f\|_2$ is satisfied. We will prove that it's satisfied for uniformly complex elliptic operators $L = -div(A\nabla)$ with bounded measurable coefficients in $\mathbb{R}^n$ in any dimension.

3.1 Introduction

Let us state the problem as it is stated in [1]. Let $A = A(x)$ be a $n \times n$ matrix of complex, L^∞ coefficients, defined on $\mathbb{R}^n$, and satisfying the ellipticity (or "accretivity") condition

$$\lambda|\xi|^2 \leq \mathcal{R}e\ A\xi \cdot \xi^* \ and\ |A\xi \cdot \zeta^*| \leq \Lambda|\xi||\zeta|, \tag{1}$$

for $\xi, \zeta \in \mathbb{C}^n$ and for some λ, Λ such that $0 < \lambda \leq \Lambda < \infty$. Here, $u \cdot v = u_1 v_1 + \cdots + u_n v_n$ and u^* is the complex conjugate of u so that $u \cdot v^*$ is the usual inner product in $\mathbb{C}^n$ and, therefore, $A\xi \cdot \zeta^* \equiv \sum_{j,k} a_{j,k}(x)\xi_k \bar{\zeta}_j$. We define a second order divergence form operator

$$Lf \equiv -div(A\nabla f). \tag{2}$$

Let $H^1(\mathbb{R}^n)$ be the Sobolev space and define the operator $\sqrt{L} : H^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ as the linear operator that satisfies $\sqrt{L}\sqrt{L} = L$. We say that $f \in H^1(\mathbb{R}^n)$ belongs to the **domain** of $\sqrt{L}$ and denote it by $f \in D(\sqrt{L})$ if

$$\|\sqrt{L}f\|_2 \leq C\|f\|_{H^1} := \|\nabla f\|_2.$$

By [3] and since our hypotheses are stable under taking adjoints, it is enough to prove that $D(\sqrt{L}) = H^1(\mathbb{R}^n)$ to conclude that

$$\|\sqrt{L}f\|_2 \sim \|\nabla f\|_2.$$

Proposition 1.

$$\sqrt{L}f = a\int_0^\infty (1+t^2L)^{-3}t^3L^2f\frac{dt}{t}, \tag{3}$$

where $a^{-1} = \int_0^\infty (1+u^2)^{-3}u^2du$.

3.2 Reduction to a Carleson measure estimate

In this section we are going to follow a $T(1)$ theorem (for square functions) argument. That means that we are going to assume that

$$\sup_Q \frac{1}{|Q|}\int_Q\int_0^{\ell(Q)} |[(1+t^2L)^{-1}tL]1(x)|^2\frac{dtdx}{t} \leq C \tag{4}$$

and we are going to prove that, up to (4), $\|\sqrt{L}f\|_2 \leq C\|\nabla f\|_2$.

3.2.1 Reduction to a quadratic estimate

Let $g \in \mathcal{C}_0^\infty$ with $\|g\|_2 = 1$, then by duality and Cauchy-Schwarz inequality,

$$\begin{aligned}
|<\sqrt{L}f, g>|^2 &= a^2\left[\int_{\mathbb{R}^n}\int_0^\infty (1+t^2L)^{-3}t^3L^2f\cdot g\frac{dtdx}{t}\right]^2\\
&= a^2\left[\int_{\mathbb{R}^n}\int_0^\infty [(1+t^2L)^{-1}tLf]\cdot[t^2L^*(1+t^2L^*)^{-2}g]\frac{dtdx}{t}\right]^2\\
&\leq a^2\int_{\mathbb{R}^n}\int_0^\infty |(1+t^2L)^{-1}tLf(x)|^2\frac{dtdx}{t}\cdot\int_{\mathbb{R}^n}\int_0^\infty |t^2L^*(1+t^2L^*)^{-2}g(x)|^2\frac{dtdx}{t}
\end{aligned}$$

We will bound the second integral in this subsection by using the **standard orthogonality argument of Littlewood-Paley theory** and treat the first one separately in the next subsection.

Pick any $\Psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with Ψ real-valued and $\int\Psi = 0$ and define Q_s as the operator of convolution with $\frac{1}{s^n}\Psi(\frac{x}{s})$ for $s > 0$ and we normalize such that

$$\int_0^\infty\int_{\mathbb{R}^n} Q_sg(x)\frac{dxds}{s} = \|g\|_2^2.$$

Lemma 2. *Let* $U_t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $t > 0$, *be a family of bounded operators with* $\|U_t\|_{op} \leq 1$. *If* $\|U_tQ_s\|_{op} \leq (inf(\frac{t}{s}, \frac{s}{t}))^\alpha$, *for some* $\alpha > 0$, *and some family* Q_s, $s > 0$ *as above, then for some constant* C *depending only on* α,

$$\int_0^\infty \int_{\mathbb{R}^n} |U_t g(x)|^2 \frac{dxdt}{t} \leq C\|g\|_2.$$

This lemma is a consequence of Schur's lemma and its proof follows the same standard argument.

We apply the lemma to $U_t = t^2L^*(1 + t^2L^*)^{-2}$ and we choose Q_s in the following manner. Let $\psi = \Delta\phi$ with $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, radial, so that, in particular, $\Psi = div\mathbf{h}$. This yields $Q_s = sdiv\mathbf{R}_s$ with $\mathbf{R}_s$ uniformly bounded.

Therefore,

$$\int_{\mathbb{R}^n} \int_0^\infty |t^2L^*(1 + t^2L^*)^{-2}g(x)|^2 \frac{dtdx}{t} \leq C\|g\|_2 = C.$$

3.2.2 The $T(1)$ Theorem argument

To simplify the notation let's denote by $\Theta_t f(x) = (1 + t^2L)^{-1}tdivAf(x)$. With this notation we are left to prove that

$$\int_0^\infty \int_{\mathbb{R}^n} |\Theta_t \nabla f(x)|^2 \frac{dxdt}{t} \leq C \int_{\mathbb{R}^n} |\nabla f(x)|^2 dx.$$

In order to benefit ourselves again with Littlewood-Paley theory we introduce P_t as a convolution operator with $\frac{1}{t^n}p(\frac{x}{t})$ where p is a smooth real-valued function supported in the unit ball of $\mathbb{R}^n$ with $\int p = 1$. By the linearity of Θ_t,

$$\begin{aligned} |\Theta_t \nabla f| &\leq |\Theta_t 1(x) \cdot (P_t^2 \nabla f)(x)| + |\Theta_t 1(x) \cdot (P_t^2 \nabla f)(x) - (\Theta_t \nabla f)(x)| \\ &\leq |\Theta_t 1(x) \cdot (P_t^2 \nabla f)(x)| + |\Theta_t 1 \cdot P_t^2 \nabla f - \Theta_t P_t^2 \nabla f| + |\Theta_t (P_t^2 - I)\nabla f| \end{aligned}$$

We apply a similar orthogonality argument as the previously described to the second and third term of the integral with $U_t P_t f(x) := \Theta_t 1(x) \cdot (P_t(P_t f))(x) - \Theta_t P_t(P_t f)(x)$ and $U_t := \Theta_t \nabla$ respectively. For the first term we apply the **Carleson's inequality** which reads as follows.

Theorem 3. *[2] Let μ be a non-negative measure, assume there exists a constant $A > 0$ such that for all $Q \in \mathbb{R}^n$ $\mu(R_Q) \leq A|Q|$ where $R_Q := Q \times (0, \ell(Q))$ then*

$$\iint_{\mathbb{R}^{n+1}_+} |T_t f|^2 d\mu(x,t) \leq C \cdot A \cdot \iint_{\mathbb{R}^{n+1}_+} |T_t f|^2 \frac{dtdx}{t}$$

We apply the above to $\mu(Q) := \int_Q \int_0^{\ell(Q)} |\Theta_t 1(x)|^2 \frac{dtdx}{t}$, and define the **Carleson measure** of μ as $A := \|\mu\|_{\mathcal{C}} = \sup_Q \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\Theta_t 1(x)|^2 \frac{dtdx}{t}$ so we get the bound

$$\begin{aligned}\int_{\mathbb{R}^n} \int_0^\infty |\Theta_t 1(x) \cdot (P_t^2 \nabla f)(x)|^2 \frac{dtdx}{t} &\leq C \|\Theta_t 1\|_{\mathcal{C}} \iint_{\mathbb{R}^{n+1}_+} |P_t^2 \nabla f|^2 \frac{dtdx}{t} \\ &\leq C \int_{\mathbb{R}^n} |\nabla f|^2 dx\end{aligned}$$

This concludes the proof up to (4).

3.3 The $T(b)$ theorem argument

Fix a cube Q, $\epsilon \in (0,1)$, a unit vector $\omega \in \mathbb{C}^n$ and define a scalar-valued function

$$b^\epsilon_{Q,\omega}(x) = (1 + (\epsilon \ell(Q))^2 L)^{-1} (\Phi_Q(x) \cdot \omega^*) \tag{5}$$

where, denoting by x_Q the center of Q,

$$\Phi_Q(x) = x - x_Q.$$

Definition 4. *Fix a cube Q and denote $\mathbb{D}(Q)$ the dyadic decomposition of the cube Q. We define the* ***averaging operator*** *on $t \in (0, \ell(Q))$ as*

$$A^Q_t f(x) = \frac{1}{|Q(x,t)|} \int_{Q(x,t)} f(y) dy \tag{6}$$

where $Q(x,t)$ is the minimal cube in $\mathbb{D}(Q)$ such that $x \in Q(x,t)$ and has side length at least t.

The remaining part of the proof is resumed in the following lemmas.

Lemma 5. *There exists an $\epsilon > 0$ depending on n, λ, Λ and a finite set W of unit vectors in $\mathbb{C}^n$ whose cardinality depends on ϵ and n, such that*

$$\begin{aligned}&\sup \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\Theta_t 1(x)|^2 \frac{dtdx}{t} \leq \\ &\qquad \leq C \sum_{\omega \in W} \sup \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\Theta_t 1(x) \cdot (A^Q_t \nabla b^\epsilon_{Q,\omega})(x)|^2 \frac{dtdx}{t}\end{aligned}$$

where C depends only on $\epsilon,\ n,\ \lambda,\ \Lambda$. The supremum is taken over all cubes Q.

Lemma 6. *For C depending only on $n,\ \lambda,\ \Lambda$ and $\epsilon > 0$, we have*

$$\int_Q \int_0^{\ell(Q)} |\Theta_t 1(x) \cdot (A_t^Q \nabla b^\epsilon_{Q,\omega})(x)|^2 \frac{dxdt}{t} \leq C|Q|. \tag{7}$$

Let's brief a sketch of the proof of both lemmas.

For lemma 5 we use a **stopping time decomposition** argument. We select cubes in $\mathbb{D}(Q)$ which are selected by being the maximal subcubes of Q satisfying at least on the following properties

$$\frac{1}{|Q'|}\int_{Q'} \mathcal{R}e(\nabla b^\epsilon_{Q,\omega}(y)\cdot\omega)dy \leq \frac{3}{4} \tag{8}$$

$$\frac{1}{|Q'|}\int_{Q'} |\nabla b^\epsilon_{Q,\omega}(y)|^2 dy \geq (4\epsilon)^{-2} \tag{9}$$

We define $\mathcal{S}'_Q$ the collection of such non-overlapping subcubes of Q and $\mathcal{S}''_Q$ the family of subcubes of Q such that are not contained or equal to any cube of $\mathcal{S}'_Q$. We also cover $\mathbb{C}^n$ with a finite family of cones $\mathcal{C}_\omega$ defined by

$$|u-(u\cdot\omega^*)\omega| \leq \epsilon|u\cdot\omega^*|$$

whose cardinality depends only on ϵ and n so that

$$\int_Q \int_0^{\ell(Q)} |\Theta_t 1(x)|^2 \frac{dtdx}{t} =$$

$$= \sum_{\omega\in W} \int_Q \int_0^{\ell(Q)} |\Theta_t 1(x)\cdot \mathbb{1}_{\mathcal{C}_\omega}(\Theta_t 1(x))|^2 \frac{dtdx}{t}.$$

Then Lemma 5 is a consequence of some geometrical arguments, and the fact that $\sum_{Q'\in\mathcal{S}'_Q} |Q'| \leq (1-\eta)|Q|$ for some $\eta \in (0,1)$. Note that η will depend on ϵ so we choose it small enough so this last condition is satisfied.

For Lemma 6 we pick a smooth cut-off function $\chi = \chi_Q$ localized on $4Q$ and equal to 1 on $2Q$ with $\|\chi\|_\infty + \ell(Q)\|\nabla\chi\|_\infty \leq C(n)$ that we introduce as follows

$$
\begin{aligned}
(7) \leq C \int_Q \int_0^{\ell(Q)} & |\Theta_t 1(x) \cdot ((A_t^Q - P_t^2)\nabla(\chi b^\epsilon_{Q,\omega})(x)|^2 \frac{dtdx}{t} + \\
& + C \int_Q \int_0^{\ell(Q)} |\Theta_t 1(x) \cdot (P_t^2 \nabla(\chi b^\epsilon_{Q,\omega}))(x)|^2 \frac{dtdx}{t} \\
\leq C \int_{\mathbb{R}^n} & |\nabla(\chi b_{Q,\omega}(x))|^2 dx + C \int_Q \int_0^{\ell(Q)} |(\Theta_t \nabla(\chi b^\epsilon_{Q,\omega}))(x)|^2 \frac{dtdx}{t} \\
\leq C|Q|. &
\end{aligned}
$$

Finally we point out that the last two inequalities are not trivial and require several technical computations that we will not include in this summary but are computed in more depth in the original paper [1, section 2 and section 5].

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4 The Tb-theorem on non-homogeneous spaces II

after F. Nazarov, S. Treil and A. Volberg [1]
A summary written by Shaoming Guo

Abstract

We discuss a Tb-theorem which extends the Tb-theorem by David, Journé and Semmes for the Calderón-Zygmund operators on $\mathbb{R}^n$ to the case of non-doubling measures.

This is the summary of the second part of the paper by Nazarov, Treil and Volberg, following the one by Polona Durcik "the Tb theorem on non-homogeneous spaces I". The theorem we will present here is the following:

Theorem 1. *Let $1 < p < \infty$ and let b_1, b_2 be two weakly accretive functions. A Calderón-Zygmund operator T is bounded on $L^p(\mu)$ if and only if the operator $b_2 T b_1$ is weakly bounded and Tb_1, T^*b_2 belong to $BMO(\mu)$.*

For the definition of "weakly accretive functions" and the choice of $BMO(\mu)$ spaces, see the summary of part I. As soon as we have the right BMO space to work with, the "only if" part of the above theorem follows from standard argument.

For the "if" part, it suffices to prove the L^2 boundedness, the L^p boundedness for general p will then follow from Calderon-Zygmund decomposition. By duality, it suffices to prove that $|\langle Tf, g\rangle| \lesssim \|f\|_2 \|g\|_2$. To estimate the inner product, we first do martingale difference decomposition for f and g, i.e. write

$$f = \sum_{Q\in\mathcal{D}} \Delta_Q^{b_1} f, g = \sum_{R\in\mathcal{D}'} \Delta_R^{b_2} g,$$

for the definition of Δ_Q^b still see the summary of part I. By linearity, we have

$$\langle Tf, g\rangle = \sum_{Q\in\mathcal{D}, R\in\mathcal{D}'} \langle T\Delta_Q^{b_1} f, \Delta_Q^{b_2} g\rangle \tag{1}$$

The whole point then is to have a good estimate for the term $\langle T\Delta_Q^{b_1} f, \Delta_Q^{b_2} g\rangle$, which we will explain now.

Case I: Q **and** R **are "far away" from each other**, by which we mean $dist(Q, R) \geq l(Q)^\gamma l(R)^{1-\gamma}$, for some $\gamma > 0$ to be chosen later.

Case II: $Q \subset R$**, but** Q **is not close to the boundary of** R, by which we mean that $dist(Q, \partial R) \geq l(Q)^\gamma l(R)^{1-\gamma}$.

Case III: singular but not essentially singular part

"singular" means if we take two cubes Q, R, w.l.o.g. say $l(Q) \leq l(R)$, then

$$dist(Q, \partial R) \leq l(Q)^\gamma l(R)^{1-\gamma}. \tag{2}$$

For the definition of being "essentially singular" see below the case IV. Roughly speaking, "singular but not essentially singular" means two cubes are close to each other, and their sizes are also comparable.

Case IV: the essentially singular part

by "essentially singular" we mean

$$eqdist(Q, \partial R) \leq l(Q)^\gamma l(R)^{1-\gamma}, l(Q) \leq 2^{-r} l(R), \tag{3}$$

for some r large number to be chosen later.

As in part I of the summary we have already seen the idea for the "regular" part-case I and case II, and part of case III, I will then focus on the rest of the "singular" part. The idea is to average over dyadic grids, "what should we do about the 'bad' ones? The surprising answer is-nothing, just ignore them! The point is that the 'bad' cubes are rare, so we don't have to worry about them."

4.1 Proof of the theorem under a stronger weak boundedness assumption

The weak boundedness assumption we will use in this section is:

$$|\langle Tb_1\chi_Q, b_2\chi_R\rangle| \lesssim \mu_Q^{1/2}\mu_R^{1/2} \tag{4}$$

for all cubes Q, R with $\frac{1}{2}l(R) \leq l(Q) \leq 2l(R)$ and $dist(Q, R) \leq \min\{l(Q), l(R)\}$.

4.1.1 random dyadic lattice

We now want to construct a random variable ξ which is uniformly distributed over the sample space consisting of all dyadic lattices, to be precise, take the standard dyadic lattice $\mathcal{D}_0$, then the element in the sample space is like $\mathcal{D}_0 + C$, where C runs through $\mathbb{R}^N$.

the construction of dyadic lattice over $\mathbb{R}$: we just need to determine the relative position of all dyadic intervals in this lattice, which is equivalent to determine for all $k \in \mathbb{Z}$ the position of one single dyadic interval of length 2^k.

Let Ω_1 be some probability space and let $x(\omega)$ be a random variable uniformly distributed over the interval $[0, 1)$. Let $\xi_j(\omega)$ be random variables satisfying $\mathcal{P}\{\xi_j = 1\} = \mathcal{P}\{\xi_j = -1\} = 1/2$. Assume also that $x(\omega), \xi_j(\omega)$ are independent. Define the random lattice as follows:
i) let $I_0(\omega) = [x(\omega) - 1, x(\omega)]$, which uniquely determines all intervals in $\mathcal{D}(\omega)$ of length 2^k with $k \leq 0$;
ii) the intervals $I_k(\omega) \in \mathcal{D}(\omega)$ of length 2^k with $k > 0$ are determined inductively: if $I_{k-1}(\omega) \in \mathcal{D}(\omega)$ is already chosen, $(I_k(\omega))_+ = I_{k-1}(\omega)$ if $\xi_k(\omega) = +1$ and $(I_k(\omega))_- = I_{k-1}(\omega)$ if $\xi_k(\omega) = -1$. In another word, in every step we extend $I_{k-1}(\omega)$ to the left if $\xi_k(\omega) = +1$ and to the right otherwise.

To get a dyadic random lattice in $\mathbb{R}^N$ we just take a product of N independent random lattices in $\mathbb{R}$. It's easy to see that the random lattice $\mathcal{D}(\omega)$ constructed in this way is uniformly distributed and satisfies

equidistribution property: for $x \in \mathbb{R}^N$, the probability that $dist(x, \partial Q) \geq \epsilon l(Q)$ is exactly $(1 - 2\epsilon)^N$ for all cubes Q.

4.1.2 bad cubes

Let $\mathcal{D}(\omega)$ and $\mathcal{D}'(\omega')$ be two independent random dyadic lattices, we call a cube $Q \in \mathcal{D}(\omega)$ "bad" if there exists a cube $R \in \mathcal{D}'(\omega')$ of lenght $l(R) \geq l(Q)$ such that Q, R are essentially singular. Otherwise we call the cube Q "good".

Lemma 2. *Let r, γ be from the definition of essentially singular pairs, then* *for any* $Q \in \mathcal{D}(\omega)$ *we have*

$$\mathcal{P}_{\omega'}\{Q \text{ is bad}\} \leq 2N \frac{2^{-r\gamma}}{1-2^{-\gamma}} \tag{5}$$

the important thing here is that when we choose r large enough, the probability will be small enough. In this sense we say that bad cubes are rare.

4.1.3 with large probability "bad" parts are small

We want to splitt the function $f = \sum_{Q\in\mathcal{D}} \Delta_Q^{b_1} f$ into two parts $f_{good} + f_{bad}$, where the bad part takes care of the bad cubes defined above

$$f_{bad} := \sum_{Q\in\mathcal{D}, Q \text{ is bad}} \Delta_Q^{b_1} f \tag{6}$$

Lemma 3. *for a given function* f *and dyadic lattice* $\mathcal{D}(\omega)$*, we have the following estimate on the mathematical expectation of* L^2 *norm of the bad parts*

$$E_{\omega'}\|f_{bad}\|_2^2 \leq 2NA^2 \frac{2^{-r\gamma}}{1-2^{-\gamma}} \|f\|_2^2, \tag{7}$$

where A *is some constant depending only on* b_1 *and* b_2*.*

We take r large enough so that $2N\frac{2^{-r\gamma}}{1-2^{-\gamma}} \leq A^{-2}2^{-8}$. Then the probability that

$$\|f_{bad}\|_2^2 \geq 4 \cdot 2^{-8}\|f\|_2^2 \tag{8}$$

can't be more than $\frac{1}{4}$, and therefore with probability $\frac{3}{4}$ we have

$$\|f_{bad}\|_2^2 \leq 2^{-6}\|f\|_2^2 \tag{9}$$

4.1.4 pulling yourself up by the hair: final proof

For an operator T, consider the cut-off operator T_ϵ, which has the trivial a priori bound $\|T_\epsilon\| \leq C_0(\epsilon, D) < \infty$, where $D := diam(supp(f) \cup supp(g))$. The point of this section is to show that the constant C_0 can be actually chosen independently of ϵ and D.

By definiton of operator norm, we could pick two L^2 normalized functions f, g such that $|\langle Tf, g\rangle| \geq \frac{1}{2}C_0(\epsilon, D)$, splitt them into good and bad parts

$$f = f_{good} + f_{bad}, g = g_{good} + g_{bad},$$

in such a way that

$$\|f_{bad}\|_2 \leq 2^{-3}\|f\|_2, \|g_{bad}\|_2 \leq 2^{-3}\|g\|_2.$$

This is always possible because we can always choose r large enough such that with a large probability the bad parts are small.

Then

$$|\langle Tf, g\rangle| \leq |\langle Tf_{good}, g_{good}\rangle| + |\langle Tf_{good}, g_{bad}\rangle| + |\langle Tf_{bad}, g\rangle|. \tag{10}$$

In case I, II and III we have seen that there exists constant C such that

$$|\langle Tf_{good}, g_{good}\rangle| \leq C\|f_{good}\|_2\|g_{good}\|_2 \leq C\|f\|_2\|g\|_2 \leq C, \tag{11}$$

By a priori bound assumption,

$$|\langle Tf_{bad}, g\rangle| \leq 2^{-3}C_0(\epsilon, D)\|f_{bad}\|_2\|g\|_2 \leq 2^{-3}C_0(\epsilon, D), \tag{12}$$

Similarly,

$$|\langle Tf_{good}, g_{bad}\rangle| \leq 2^{-3}C_0(\epsilon, D)\|f_{good}\|_2\|g_{bad}\|_2 \leq 2^{-3}C_0(\epsilon, D). \tag{13}$$

Notice that we have choosen f, g s.t. $|\langle Tf, g\rangle| \geq \frac{1}{2}C_0(\epsilon, D)$, we then get

$$C_0(\epsilon, D) \leq |\langle Tf, g\rangle| \leq C + 2 \cdot \frac{1}{8}C_0(\epsilon, D). \tag{14}$$

Therefore $C_0(\epsilon, D) \leq 4C$, which is independent of ϵ and D, then we are done.

4.2 proof of the full Tb theorem

4.2.1 weak boundedness on rectangular boxes

In this subsection let's first consider a special case, which has the following weak boundedness assumption:

$$|\langle T\chi_Q b_1, \chi_Q b_2\rangle| \lesssim \mu(Q), \forall \text{ rectangular boxes } Q \tag{15}$$

The difference from the previous section is that now we are not allowed to control $|\langle Tf_{good}, g_{good}\rangle|$ by weak boundedness assumption any more, because the weak boundedness assumption is of the form of $Q = R$, but now we have infinitely many terms with different Q and R of comparable size such that $Q \cap R \neq \emptyset$.

The idea to handle this term is the same as what we did for the essentially singular part, we will throw more "bad" parts from f_{good}, g_{good}, and try to get an estimate of the form

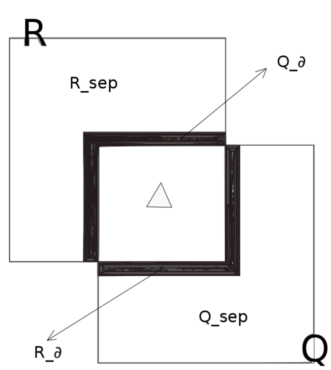
$$|\langle T f_{good}, g_{good} \rangle| \leq \frac{1}{4}\|T\| + C. \tag{16}$$

To estimate $|\langle T f_{good}, g_{good} \rangle|$ it's enough to estimate the sum

$$\sum_{Q,R} |\langle T\Delta_Q^{b_1} f, \Delta_R^{b_2} g \rangle|, \tag{17}$$

over all cubes of comparable size $2^{-r} l(Q) \leq l(R) \leq 2^r l(Q)$.

The cancellation from $\Delta_Q^{b_1} f$ will not play any role here, so we just write it as linear combination of characteristic functions, then it suffices to estimate terms of the form $|\langle T\chi_Q b_1, \chi_R b_2 \rangle|$, where again Q, R have comparable size.



In the above picture, the dark part $Q_\partial \cup R_\partial$ will be the "bad" part that we will throw away, this is always possible provided that we choose the thickness of the dark part to be small enough. For the rest, either we know that R_{sep}, Q_{sep}, Δ are seperated, for which we have good estimate, or we will use the weak boundedness assumption to estimate the term $\langle T\chi_\Delta, \chi_\Delta\rangle$, as Δ is a rectangular box.

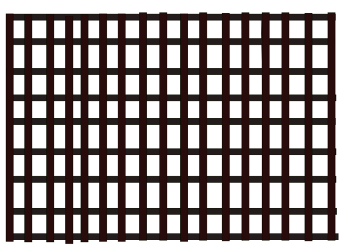
4.2.2 proof of the full Tb theorem

In this section we will use the following weak boundedness assumption:

$$|\langle T\chi_Q b_1, \chi_Q b_2\rangle| \lesssim \mu(Q), \forall \text{ cubes } Q \tag{18}$$

The only difference with the last section "weak boundedness on rectangular boxes" is that, under the above weak boundedness assumption, we don't have an estimate for the term $\langle T\chi_\Delta, \chi_\Delta\rangle$ directly as Δ is just a rectangular box instead of a cube.

The way out is pretty easy, although rectangular boxes are not cubes, they can still be covered by cubes, there's some problem near the boundary of the rectangular boxes, but again ignore them because those cubes are really rare!



In the above picture, the largest rectangular box is $\Delta = Q \cap R$, the dark part is the thin boundary of a dyadic lattice of fixed width, we can take the thickness to be so small that f, g restricted on this part will have "no" contribution. The rest is just a union of cubes, for two cubes in this collection, they are either seperated, for which we have good estimate(with constant apparently depending on the thickness of the dark part), or they are the same, for which we could use weak boundedness assumption!

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5 Sharp weighted estimates for dyadic shifts and the A_2 conjecture

after Tuomas Hytönen, Carlos Pérez, Sergei Treil, Alexander Volberg [1]
A summary written by Timo Hänninen

Abstract

We outline the proof of the A_2 theorem given in the above-mentioned paper.

5.1 Introduction

A *weight* is a strictly positive locally integrable function on $\mathbb{R}^d$. The A_2 *characteristic* $[w]_{A_2}$ of a weight w is defined by $[w]_{A_2} := \sup_{\text{Q is a cube}} \langle w \rangle_Q \langle \frac{1}{w} \rangle_Q$. The A_2 *class* consists of all the weights w such that $[w]_{A_2} < \infty$.

Theorem 1 (A_2 theorem)**.** *For each Calderón-Zygmund operator T there exists a constant C_T such that*

$$\|T\|_{L^2(w)\to L^2(w)} \leq C_T [w]_{A_2} \quad \textit{for all weights } w \in A_2.$$

After many intermediate results by others, the A_2 theorem in full generality was first proven by Hytönen [2]. The paper [1], which we summarize here, gives a simplified proof of the A_2 theorem. For lecture notes on the proof, see [4], and for a survey on further simplications of the proof, see [3].

The first step in the proof is to represent each Calderón-Zygmund operator as a series of dyadic shifts and dyadic paraproducts averaged over randomized dyadic systems. A *dyadic shift* $S^{ij}_{\mathcal{D}}$ associated with non-negative integers i and j and a collection of dyadic cubes $\mathcal{D}$ is an operator of the form

$$S^{ij}_{\mathcal{D}} f := \sum_{K\in\mathcal{D}} A_K := \sum_{K\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}, J\in\mathcal{D}, I\subseteq K, J\subseteq K \\ \ell(I)=2^{-i}\ell(K),\, \ell(J)=2^{-j}\ell(K)}} a_{IJK} \langle f, h_I \rangle h_J$$

for Haar functions h_I, which are L^2-normalized, and for some coefficients a_{IJK} that satisfy the size condition $|a_{IJK}| \leq \frac{\sqrt{|I||J|}}{|K|}$. The point of the size condition is to ensure that $|A_K f| \leq 1_K \langle |f| \rangle_K$. We say that the shift $S^{ij}_{\mathcal{D}}$ is *generalized* if we allow some of the functions h_I to be of the form $h^0_I = |I|^{-1/2} 1_I$. We say the shift $S^{ij}_{\mathcal{D}}$ has the *complexity* $\kappa := \max\{i, j\} + 1$.

There are 2^d possible choices for the dyadic parent of a cube. By choosing the parents at random, we can define the randomized dyadic systems $\mathcal{D}^\omega$.

Theorem 2 (Dyadic representation theorem)**.** *Let T be a Calderón-Zygmund operator with the Hölder exponent α. Then there exist dyadic shifts $S^{ij}_{\mathcal{D}^\omega}$ and a constant C_T such that*

$$\langle g, Tf\rangle = \mathbb{E}_\omega\left(C_T \sum_{i\geq 0, j\geq 0} 2^{-(i+j)/(2\alpha)}\langle g, S^{ij}_{\mathcal{D}^\omega} f\rangle + \langle g, \Pi^{\mathcal{D}^\omega}_{T1} f\rangle + \langle g, (\Pi^{\mathcal{D}^\omega}_{T^*1})^* f\rangle \right)$$

for all, say, $f \in C^1_c(\mathbb{R}^d)$ and $g \in C^1_c(\mathbb{R}^d)$. Moreover, the constant C_T depends only on the dimension, on the Hölder exponent, on the constant in the weak boundedness property, and on the constants in the standard estimates for the kernel.

The second step is a Sawyer-type characterization for the boundedness of generalized dyadic shifts from a weighted L^2 space to another. Let u and w be weights. For the *dual weight* $\sigma := u^{-1}$ of the weight u we have that

$$\|T\|_{L^2(u)\to L^2(w)} = \|T(\,\cdot\,\sigma)\|_{L^2(\sigma)\to L^2(w)}$$

and that the weights w and σ are interchanged by taking the adjoint: The formal adjoint of the operator $T(\,\cdot\,\sigma) : L^2(\sigma) \to L^2(w)$ is the operator $T^*(\,\cdot\, w) : L^2(w) \to L^2(\sigma)$. Note that in the special case of one weight we have $u = w$, and hence $\sigma = w^{-1}$.

Let $S_{\mathcal{D}}$ be a generalized dyadic shift with complexity κ associated with a dyadic system $\mathcal{D}$. Let $\mathcal{D}_n$ denote the collection of all the dyadic cubes with side length 2^{-n}. We may separate the dyadic length scales by picking every κth dyadic length scale: For each integer k with $0 \leq k \leq \kappa - 1$ we define $\mathcal{D}_{k \bmod \kappa} := \bigcup_{n\in\mathbb{Z}} \mathcal{D}_{k+n\kappa}$.

We say that each dyadic shift $S_{\mathcal{D}_{k \bmod \kappa}}$ has its *scales separated*. Note that $S_{\mathcal{D}} = \sum_{k=0}^{\kappa-1} S_{\mathcal{D}_{k \bmod \kappa}}$. The point of separating scales is that A_K is constant on K' whenever $K' \subsetneq K$ and $\ell(K') = 2^{-\kappa}\ell(K)$. In the series of the dyadic representation theorem, to sum the termwise estimates we need that they decay in the complexity fast enough. The motto is that the estimates for dyadic shifts with scales separated are independent of the complexity.

Theorem 3 (Two weight testing conditions for generalized dyadic shifts)**.** *Let S be a generalized dyadic shift with scales separated. Let w and σ be weights. Suppose that for some constants $[w, \sigma]_{A_2}$, $\mathfrak{S}$, and $\mathfrak{S}^*$, we have that*

$$\langle w\rangle_Q \langle\sigma\rangle_Q \leq [w, \sigma]_{A_2} \quad \textit{for all dyadic cubes } Q\textit{, and, moreover, that}$$

$$\|1_Q S(1_Q\sigma)\|_{L^2(w)} \le \mathfrak{S}\sigma(Q)^{1/2} \quad \textit{and} \quad \|1_Q S^*(1_Q w)\|_{L^2(\sigma)} \le \mathfrak{S}^* w(Q)^{1/2}$$

for all dyadic cubes Q. Then for some absolute constant C we have

$$\|S(f\sigma)\|_{L^2(w)} \le C\left(\mathfrak{S} + \mathfrak{S}^* + [w,\sigma]_{A_2}^{1/2}\right)\|f\|_{L^2(\sigma)} \quad \textit{for all } f \in L^2(\sigma).$$

The third step is to verify the testing conditions, step which we shall not discuss further in this summary.

Theorem 4 (Verification of the testing conditions)**.** *Let S be a generalized dyadic shift with scales separated. Suppose that S is bounded on L^2 with the operator norm at most one. Let w be a weight. Let $\sigma := w^{-1}$ be the dual weight of w. Then for some constant C_d we have*

$$\|1_Q S(1_Q\sigma)\|_{L^2(w)} \le C_d[w,\sigma]_{A_2}\sigma(Q)^{1/2} \quad \textit{for all dyadic cubes } Q.$$

5.2 Proof of the dyadic representation theorem

In this section we outline the proof of Theorem 2. We fix a non-negative integer r and a real number γ with $0 < \gamma < 1$. We say that a dyadic cube $I \in \mathcal{D}$ is *good* if the boundary of every much bigger cube lies far away from it:

$$\operatorname{dist}(I, \partial J) > (\ell(J)/\ell(I))^{(1-\gamma)}\,\ell(I) \quad \text{for every } J \in \mathcal{D} \text{ with } \ell(J) \ge 2^r \ell(I).$$

Let $f \in L^2$ and $g \in L^2$. By expanding the functions in the Haar basis and splitting the summation, we obtain

$$\langle g, Tf\rangle = \sum_{\substack{I\in\mathcal{D},\ J\in\mathcal{D}: \\ \text{the cube with} \\ \text{smaller side} \\ \text{length is good}}} \langle g, h_J\rangle\langle h_J, Th_I\rangle\langle f, h_I\rangle + \sum_{\substack{I\in\mathcal{D}, J\in\mathcal{D}: \\ \text{the cube with} \\ \text{smaller side} \\ \text{length is bad}}} \langle g, h_J\rangle\langle h_J, Th_I\rangle\langle f, h_I\rangle.$$

Let $\langle g, Tf\rangle =: \langle g, Tf\rangle^{\mathcal{D}}_{\text{good}} + \langle g, Tf\rangle^{\mathcal{D}}_{\text{bad}}$. The point of averaging over random dyadic systems is that on average the bad part is comparable to the good part.

Proposition 5. *We have*

$$\mathbb{E}_\omega\langle g, Tf\rangle^{\mathcal{D}^\omega}_{bad} = C_{r,\gamma,d}\,\mathbb{E}_\omega\langle g, Tf\rangle^{\mathcal{D}^\omega}_{good} \quad \textit{for all } f \in C^1_c(\mathbb{R}^d) \textit{ and } g \in C^1_c(\mathbb{R}^d).$$

Hence it suffices to estimate the good part. Let $I \vee J$ denote the minimal dyadic cube that contains both I and J. By rearranging the summation, we have

$$\langle g, Tf\rangle^{\mathcal{D}}_{\text{good}} = \sum_{i\geq j\geq 0} \langle g, (\sum_{K\in\mathcal{D}} \sum_{\substack{I\in\mathcal{D}, J\in\mathcal{D}: I\vee J=K,\\ \ell(I)=2^{-i}\ell(K), \ell(J)=2^{-j}\ell(K),\\ \text{and I is good}}} \langle h_J, Th_I\rangle\langle f, h_I\rangle h_J)\rangle$$

$$+ (f \text{ and } g \text{ interchanged, } T \text{ and } T^* \text{ interchanged}).$$

We note that the operator in the round brackets has the form of a dyadic shift. Hence it remains to prove that the size condition

$$|\langle h_J, Th_I\rangle| \leq C 2^{-\alpha(i+j)/2} \frac{\sqrt{|I||J|}}{|I\vee J|}$$

holds whenever I is good, $\ell(I) \leq \ell(J)$, $\ell(I) = 2^{-i}\ell(K)$, and $\ell(J) = 2^{-j}\ell(K)$. There are three alternative cases: $I \subsetneq J$, $I = J$, or $I \cap J = \emptyset$. The case '$I = J$' is estimated by the weak boundedness property. The case '$I \cap J = \emptyset$' and, after separating the dyadic paraproducts, the case '$I \subsetneq J$' are estimated by the standard estimates. The point of a good cube is that it stays away from the boundaries of all other (much bigger) cubes. This, after quantifying 'away', has two consequences: In the case '$I \subsetneq J$ ' we have a lower bound for $\text{dist}(I, \partial J)$ and in the case '$I \cap J = \emptyset$' we have an upper bound for $\ell(I \vee J)$. These bounds yield the exponential decay in $(i + j)$.

5.3 Proof of the two weight testing conditions

In this section we give a proof of Theorem 3, whose statement and proof vary from the corresponding original theorem [1, Theorem 3.4.] in that we have separated scales of dyadic shifts and of Haar projections. Let $S := S_{\mathcal{D}_{k \bmod \kappa}}$ be a generalized dyadic shift with scales separated and with complexity κ. Throughout this section it is understood that all dyadic cubes that we are considering belong to the collection $\mathcal{D}_{k \bmod \kappa}$. With this convention, the operator A_K has the following two crucial properties:

- $A_K f = 1_K A_K(1_K f)$
- $A_K f$ is constant on K' whenvever $K' \subsetneq K$.

Let $f \in L^2(\sigma)$. We use the notations $\sigma(Q) := \int_Q \sigma$ and $\langle f\rangle^\sigma_Q := \frac{1}{\sigma(Q)}\int_Q f\sigma$. We define *weighted Haar projections with scales separated* by

$$D^\sigma_Q f := \sum_{Q_i: Q_i\subseteq Q \text{ and } \ell(Q_i)=2^{-\kappa}\ell(Q)} 1_{Q_i}\langle f\rangle^\sigma_{Q_i} - 1_Q\langle f\rangle^\sigma_Q.$$

The operator D_Q^σ has the following three crucial properties:

- $D_Q^\sigma f = 1_Q D_Q^\sigma(1_Q f)$
- $\langle 1_Q, \sigma D_Q^\sigma f\rangle = 0$
- $D_Q^\sigma f$ is constant on Q' whenever $Q' \subsetneq Q$.

We may assume that f can be expanded as $f = \sum_Q D_Q^\sigma f$. We deal with $g \in L^2(w)$ similarly. Let us consider the dual pairing

$$\langle wg, S(\sigma f)\rangle = \sum_{Q,R,K} \langle w D_R^w g, A_K(\sigma D_Q^\sigma f)\rangle.$$

By using the crucial properties of the operators A_K, D_Q^σ, and D_R^w, we observe that $\langle w D_R^w g, A_K(\sigma D_Q^\sigma f)\rangle = 0$ unless we have one of the following four alternative cases: $K \subseteq Q \subsetneq R$, $K \subseteq R \subsetneq Q$, $Q = R = K$, or $K \subsetneq Q = R$. For further convenience, we define the operator $S_Q := \sum_{K \subseteq Q} A_K$.

First we check the case '$K \subseteq Q \subsetneq R$'. By the fact that $D_R^w g$ are cancellative on R and constant on the proper subcubes of R, by the expansion $g = \sum_R D_R^w g$, by the fact that $D_Q^\sigma D_Q^\sigma = D_Q^\sigma$ and $(D_Q^\sigma)^* = D_Q^\sigma$, by the Cauchy–Schwarz inequality applied twice, first to the weighted integral and then to the summation over Q, and by Pythagoras' theorem, we have

$$\begin{aligned}
|\sum_{Q,R,K : K \subseteq Q \subsetneq R} \langle w D_R^w g, A_K(\sigma D_Q^\sigma f)\rangle| &= |\sum_Q \langle \sum_R D_R^w g\rangle_Q^w \langle D_Q^\sigma S_Q^*(1_Q w), \sigma D_Q^\sigma f\rangle| \\
&\le \left(\sum_Q (\langle |g|\rangle_Q^w)^2 \|D_Q^\sigma S_Q^*(w)\|_{L^2(\sigma)}^2 \right)^{1/2} \|f\|_{L^2(\sigma)}.
\end{aligned}$$

We use the dyadic Carleson embedding theorem to estimate the quantity in the round brackets. The key observation is that $D_{Q'}^\sigma S_{Q'}^*(w) = D_{Q'}^\sigma S_Q^*(w)$ whenever $Q' \subseteq Q$, because $1_{Q'} S_Q^*(w) = 1_{Q'} S_{Q'}^*(w) + 1_{Q'} \sum_{K : Q \supseteq K \supsetneq Q'} A_K^*$, where the second term is constant on Q'. Then the condition in the dyadic Carleson embedding theorem is checked by using Pythagoras' theorem, the expansion $1_Q S_Q^*(w) = 1_Q \langle S_Q^*(w)\rangle_Q^\sigma + \sum_{Q' \subseteq Q} D_{Q'}^\sigma S_Q^*(w)$, and the direct testing condition. The case '$K \subseteq R \subsetneq Q$' is checked in a similar way.

Next we check the case '$K = R = Q$'. By the estimate $|A_Q f| \le \frac{1}{|Q|}\|f 1_Q\|_{L^1}$, by the Cauchy–Schwarz inequality applied to each of the weighted integrals, and by the definition of the joint A_2 characteristic $[w,\sigma]_{A_2}$, we have

$$\begin{aligned}
&|\langle w D_Q^w g, A_Q(\sigma D_Q^\sigma f)\rangle| \le \frac{1}{|Q|}\|D_Q^w g\|_{L^1(w)} \|D_Q^\sigma f\|_{L^1(\sigma)} \\
&\le \frac{w(Q)^{1/2}\sigma(Q)^{1/2}}{|Q|} \|D_Q^w g\|_{L^2(w)} \|D_Q^\sigma f\|_{L^2(\sigma)} \le [w,\sigma]_{A_2}^{1/2} \|D_Q^w g\|_{L^2(w)} \|D_Q^\sigma f\|_{L^2(\sigma)}.
\end{aligned}$$

By summing over the dyadic cubes Q, applying the Cauchy–Schwarz inequality to the summation, and using Pythagoras' theorem, we conclude the case.

Next we check the case '$K \subsetneq Q = R$'. Let Q_k be the maximal, and hence pairwise disjoint, dyadic cubes strictly contained in Q. By expanding $\sum_{K \subsetneq Q} A_K = \sum_{Q_k} S_{Q_k}$, $D_Q^\sigma f = \sum_{Q_i} \langle f \rangle_{Q_i}^\sigma 1_{Q_i}$, and $D_Q^w g = \sum_{Q_j} \langle g \rangle_{Q_j}^w 1_{Q_j}$, using the fact that Q_i are pairwise disjoint, applying the Cauchy–Schwarz inequality to the weighted integral, using the direct testing condition (or, alternatively, the dual testing condition) and the joint A_2 condition to obtain the estimate $\|S_{Q_k}(\sigma)\|_{L^2(w)} \leq (\mathfrak{S} + [w,\sigma]_{A_2}^{1/2}) \sigma(Q_k)^{1/2}$, applying the Cauchy–Schwarz inequality to the summation over Q_k, and using the definition of the integral, we obtain

$$
\begin{aligned}
&|\langle w D_Q^w g, \sum_{K \subsetneq Q} A_K(\sigma D_Q^\sigma f) \rangle| \leq \sum_{Q_k} |\langle D_Q^w g \rangle_{Q_k}^w| \, |\langle D_Q^\sigma f \rangle_{Q_k}^\sigma| \|1_{Q_k}\|_{L^2(w)} \|S_{Q_k}(\sigma)\|_{L^2(w)} \\
&\leq \left(\mathfrak{S} + [w,\sigma]_{A_2}^{1/2} \right) \|D_Q^w g\|_{L^2(w)} \|D_Q^\sigma f\|_{L^2(\sigma)}.
\end{aligned}
$$

By summing over the dyadic cubes Q, applying the Cauchy–Schwarz inequality to the summation, and using Pythagoras' theorem, we conclude the case.

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6 Wavelet method for Cauchy integral and Kato conjecture

after Ph. Tchamitchian [6], P. Auscher and Ph. Tchamitchian [1]
A summary written by Yi Huang

Abstract

We give a brief survey on the wavelet methods used in re-proving the boundedness of Cauchy integral along lipschitz curves [6] and in re-solving the one dimension Kato conjecture [1]. If time permits, we shall also connect the wavelet objects with the T(b) scheme.

6.1 Introduction

Let the lipschitz curve Γ be the graph of a real-valued function A, which is assumed to be defined on $\mathbb{R}$, differentiable almost everywhere and with A' bounded. The Cauchy integral along the lipschitz curve Γ is defined as

$$\mathcal{C}_A f(x) = p.v.\frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(y)}{x-y+i(A(x)-A(y))}(1+iA'(y))dy. \tag{1}$$

The celebrated theorem of Coifman-McIntosh-Meyer [3] affirms that

$$\underline{\mathcal{C}_A \text{ is bounded on } L^2(\mathbb{R})}.$$

In 1964, A. P. Calderón [2] used for the first time the following transform

$$\forall\, f \in L^2(\mathbb{R}),\ f = \int_{\mathbb{R}_+}\int_{\mathbb{R}} \langle f, \psi_{a,b}\rangle \psi_{a,b}\frac{dbda}{a}, \tag{2}$$

where $\psi_{a,b}(x) = a^{1/2}\psi(ax-b)$ and $\psi \in L^2(\mathbb{R})$ with

$$\int_{\mathbb{R}_+} |\widehat{\psi}(a\xi)|^2 \frac{da}{a} = 2\pi,\ \forall\, \xi \neq 0.$$

This kind of decomposition has been employed to study the boundedness of singular integral operators of Calderón-Zygmund type, in particular, the Cauchy integral on lipschitz curves as in [3]. The discrete version of (2) is

$$f = \sum_{j,k} \langle f, \psi_{j,k}\rangle \psi_{j,k}, \tag{3}$$

where $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbb{Z}$, and ψ is properly chosen.

Just like its continuous version, (3) can also be used to study the boundedness of singular integral operators, say for example, the re-proof in [6] for Cauchy integral on lipschitz curves. Moreover, (3) enables us to construct the bases of $L^2(\mathbb{R})$ and many other function spaces, and this is the main idea taken in [1] to re-solve the one dimension Kato conjecture.

6.2 Wavelets and Cauchy integral on lipschitz curves

Let $b(x) = 1 + iA'(x)$, and define the bilinear symmetric form B by

$$B(f, g) = \int_{\mathbb{R}} f(x)g(x)b(x)dx.$$

The boundedness of $\mathcal{C}_A$ on $L^2(\mathbb{R})$ arrives as a corollary of the following result.

Theorem 1 ([6]). *There exist a family of b-wavelets $\Theta_{j,k}$, $j, k \in \mathbb{Z}$, and two constants $C, \alpha > 0$, such that the following properties hold true*

$$|\Theta_{j,k}(x)| \leq C2^{j/2}e^{-\alpha|2^j x - k|}, \tag{4}$$

$$|\Theta'_{j,k}(x)| \leq C2^{3j/2}e^{-\alpha|2^j x - k|}, \tag{5}$$

$$\int_{\mathbb{R}} \Theta_{j,k}(x)b(x)dx = 0, \tag{6}$$

$$B(\Theta_{j,k}, \Theta_{j',k'}) = \delta_{(j,k),(j',k')}, \tag{7}$$

$$\forall\, f \in L^2(\mathbb{R}),\ \int_{\mathbb{R}} |f|^2 \simeq \sum_{j,k} |B(f, \Theta_{j,k})|^2. \tag{8}$$

Here, the expression "family of b-wavelets" means that the wavelets are adapted to b, and that there exists $\Theta(x)$ such that $\Theta_{j,k}(x) = 2^{j/2}\Theta(2^j x - k)$.

6.3 Wavelets and Kato conjecture in one dimension

Let the complex-valued function $a(x) \in L^\infty(\mathbb{R})$, with Re $a(x) \geq 1$ almost everywhere. Denote by $D = -i\, d/dx$ the differentiation operator and by A the pointwise multiplication by $a(x)$. Consider the sesquilinear form

$$J(f, g) = \int_{\mathbb{R}} a(x)Df(x)\overline{Dg(x)}dx,$$

which is defined on the Sobolev space $H^1(\mathbb{R})$. Thus we have

$$\text{Re } J(f,f) \geq \|Df\|_2^2.$$

It was shown by T. Kato in [4] that the form J defines a maximal accretive operator T, which we write as D^*AD, where the domain $\mathbf{D}(T)$ is the largest subspace in $H^1(\mathbb{R})$ and is consisted by those functions f such that

$$J(f,g) = \langle Tf, g\rangle, \ \forall\, g \in H^1(\mathbb{R}).$$

The square root of T is then defined by the functional calculus of T. Kato, and the conjecture of Kato is to determinate the exact domain of $T^{1/2}$.

The difficulty comes from the fact that $a(x)$ is not regular. Indeed, $\mathbf{D}(T)$ coincides with the set of functions $f \in H^1(\mathbb{R})$ such that $a(x)Df(x) \in H^1(\mathbb{R})$. But this space is not classical and in particular is not realized in the scale of Sobolev spaces $H^{1+\epsilon}(\mathbb{R})$, $\epsilon > 0$. The Kato conjecture (in one dimension) was solved in 1982 by R. Coifman, A. McIntosh and Y. Meyer [3]:

<u>The domain of $T^{1/2}$ is the space $H^1(\mathbb{R})$ (with the equivalence of norms)</u>.

Basing on the wavelet method in [6] in reproving the boundedness of $\mathcal{C}_A$, the authors in [1] characterized $\mathbf{D}(T)$ by an appropriate wavelet basis.

Theorem 2 ([1])**.** *Denote by Λ the set of dyadic intervals of $\mathbb{R}$. There exists a family of lipschitz functions $\tau_\lambda(x)$, $\lambda \in \Lambda$, which belong to $\mathbf{D}(T)$ and form an unconditional base of each of the spaces $L^2(\mathbb{R})$, $H^1(\mathbb{R})$ and $\mathbf{D}(T)$.*

Moreover, the respective membership of $\sum \alpha_\lambda \tau_\lambda(x)$ in each of these spaces are characterized by $\{\alpha_\lambda\} \in l^2(\Lambda)$, $\{\alpha_\lambda\} \in l^2(\Lambda, \omega)$ and $\{\alpha_\lambda\} \in l^2(\Lambda, \omega^2)$. Here, ω is a certain positive weight defined on Λ.

Once the above theorem is established, we arrive at the solution of one dimension Kato conjecture by using a result of J.L. Lions.

Lemma 3 ([4])**.** *If T is an operator with domain $\mathbf{D}(T)$ in a Hilbert space H, and T is maximal accretive, then the domain of the square root of T is the space of complex interpolation at the mid-point between $\mathbf{D}(T)$ and H.*

To complete the proof of Theorem 2, the following result is needed, and it is essentially in the same spirit of [6] (see Theorem 1 above).

Lemma 4 ([1])**.** *There exist two constants $C, \gamma > 0$ and a family of complex-valued C^2 functions $\{\theta_\lambda(x)\}_{\lambda\in\Lambda}$, such that if $\lambda = [k2^{-j}, (k+1)2^{-j})$, $k, j \in \mathbb{Z}$, we have*

$$|\theta_\lambda(x)| \leq C2^{j/2}e^{-\gamma|2^j x-k|}, \tag{9}$$

$$|D\theta_\lambda(x)| \leq C2^{3j/2}e^{-\gamma|2^j x-k|}, \tag{10}$$

$$|D^2\theta_\lambda(x)| \leq C2^{5j/2}e^{-\gamma|2^j x-k|}, \tag{11}$$

$$\int_{\mathbb{R}} \theta_\lambda(x)b(x)dx = 0, \tag{12}$$

$$\int_{\mathbb{R}} x\theta_\lambda(x)b(x)dx = 0, \tag{13}$$

where $b(x) = 1/a(x)$, $a(x)$, D and Λ are as before. Moreover, we have

$$\int_{\mathbb{R}} \theta_\lambda(x)\theta_\mu(x)b(x)dx = \delta_{\lambda,\mu},\ \lambda \in \Lambda,\ \mu \in \Lambda. \tag{14}$$

The collection of all these functions constitutes an unconditional base of $L^2(\mathbb{R})$: all the function $f(x)$ can be written uniquely as $f(x) = \sum \alpha_\lambda\theta_\lambda(x)$ with $\{\alpha_\lambda\} \in l^2(\Lambda)$. Moreover, $\|f\|_2$ and $(\sum|\alpha_\lambda|^2)^{1/2}$ are two equivalent norms. In the end, for all $\lambda \in \Lambda$,

$$\alpha_\lambda = \int_{\mathbb{R}} f(x)\theta_\lambda(x)b(x)dx. \tag{15}$$

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7 A new proof of Moser's Parabolic Harnack Inequality using the old ideas of Nash

after E. B. Fabes and D. W. Stroock [5]
A summary written by Sukjung Hwang

Abstract

We revisit Nash's idea to show a Harnack inequality, quantitative relations between the values of solutions at different points, of second order parabolic differential equations in divergence form.

7.1 Introduction

We introduce second order elliptic and parabolic operators of the form

$$L^e = \sum_{i,j=1}^{n} D_{x_i}\left(a^e_{ij}(x) D_{x_j}\right) \tag{1}$$

$$L = \sum_{i,j=1}^{n} D_{x_i}\left(a_{ij}(t,x) D_{x_j}\right) - D_t \tag{2}$$

where t is a real number and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Basic assumptions on the matrix are symmetry, i.e. $a^e_{ij} = a^e_{ji}$ and $a_{ij} = a_{ji}$, and the existence of a number $\lambda \in (0,1]$ such that for all $(t,x) \in \mathbb{R}^{n+1}$ and all nonzero $\xi \in \mathbb{R}^n$

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a^e_{ij}(x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2,$$

and

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2, \tag{3}$$

which is so called uniform ellipticity and boundeness of the coefficient matrix.

Before DeGiorgi-Nash-Moser, the regularity theory in partial differential equations is studied based on perturbation type of arguments such as Schauder's estimate obtaining a priori estimates depends on the smootheness of coefficients and the boundary. Assuming only measurability, uniform ellipticity, and boundedness of the coefficients, DeGiorgi (1957,[4]) gives the

Hölder continuity of weak solutions satisfying $L^e u = 0$ by taking iteration that a certain integral quantity of u decays in between two different sized balls, yielding a proper relation of the oscillations like Lemma 1. Later, Moser (1961 [6] for L^e (1) and 1964 [7] for L) introduces another iteration scheme to control the power of L_p norm of weak solutions leading to a Harnack inequality. Those two powerful techniques are adopted and extended by many mathematicians (Aronson, Chen, DiBenedetto, Kurihara, Ladyženskaja, Serrin, Trudinger, Ural'ceva, etc.).

In 1958, Nash [8] approaches the regularity theory of the parabolic operator L with physical intuition and regards the elliptic operator L^e as a specialization typically the steady state of parabolic equations. However, the complexity and difficulty of his proof does not permit one to sharpen his results easily. In the paper [5], a Harnack inequality is given by combining Nash's idea and techniques developed later in a simper way.

7.2 The upper and lower bounds

We may assume that the matrix $a(t,x) \equiv a_{ij}(t,x)$ of the parabolic operator L (2) is smooth; however, we emphasize that all quantitative estimates (a priori constants) are only allowed to depend on n and λ, from (3). For $x, y, \xi \in \mathbb{R}^n$ and $t, s, r \in \mathbb{R}$, let $\Gamma(t,x;s,y) \equiv \Gamma_a(t,x;s,y)$ denote the fundamental solution of the parabolic operator L, in (2). The purpose of this paper is to use the ideas of Nash [8] to obtain the following estimates: for $s < t$

$$\frac{\exp\left\{-\frac{C|x-y|^2}{t-s}\right\}}{C(t-s)^{n/2}} \leq \Gamma(t,x;s,y) \leq \frac{C\exp\left\{-\frac{|x-y|^2}{C(t-s)}\right\}}{(t-s)^{n/2}} \tag{4}$$

where C depends only on n and λ.

The upper bound (described as 'the moment bound' in Nash's paper, Part I on [8]) is given applying techniques by Davies [3]. First, define

$$f_t(x) = \exp(-\psi(x)) \int f(y)\Gamma(t,x;0,y)\exp(\psi(y))\,dy,$$

where $f \in S(\mathbb{R}^n;(0,\infty))$, a positive function from the Schwartz test function space, and $\psi(x) = \alpha \cdot x$ with a fixed element $\alpha \in \mathbb{R}^n$. From various inequalities, one provides that for any $p \in [1,\infty)$, $t \geq 0$, and some $\epsilon > 0$

$$\frac{d}{dt}\|f_t\|_{2p} \leq -\frac{\epsilon}{2p}\|f_t\|_{2p}^{1+4p/n}\|f_t\|_p^{-4p/n} + \frac{|\alpha|^2 p}{\lambda}\|f_t\|_{2p}.$$

For each $\delta > 0$, the above inequality implies that there is a constant $K = K(\epsilon, \delta) < \infty$ such that for $t \geq 0$

$$\|f_t\|_{2p} \leq (Kp^2)^{n/4p} \|f_t\|_p e^{\delta\alpha^2 t/\lambda p} t^{n(1-p)/4p}.$$

In particular, we also obtain for $t \geq 0$

$$\|f_t\|_2 \leq e^{|\alpha|^2 t/\lambda} \|f\|_2.$$

By setting $p_k = 2^k$ and $w_k = \max\{s^{n(p_k-2)/4p_k} \|f_t\|_{p_k} : 0 \leq s \leq t\}$, if $\|f\|_2 = 1$ and $u_1(t) \leq e^{\alpha^2 t/\lambda}$, then we have

$$\frac{w_{k+1}(t)}{w_k(t)} \leq (4^k K)^{n/4\cdot 2^k} \exp\left(\frac{\delta\alpha^2 t}{\lambda 2^k}\right)$$

and there is a constant $C < \infty$, depending only on n, λ and $\delta > 0$, such that

$$\sup_k w_k(t) \leq C \exp\left[\frac{(1+\delta)\alpha t}{\lambda}\right].$$

Therefore, by taking $k \to \infty$, it follows

$$\|f_t\|_\infty \leq C t^{-n/4} \exp\left[\frac{(1+\delta)\alpha t}{\lambda}\right].$$

By using the adjoint operator and the duality, it allows us to have

$$\Gamma_a(2t, x; 0, y) \leq \frac{C}{t^{n/2}} \exp\left(4|\alpha|^2 t/\lambda + \alpha \cdot (x-y)\right)$$

and the upper bound, the right-hand side of (4), is provided by choosing $\alpha = \frac{\lambda}{8t}(y-x)$.

The lower bound, the left-hand side of (4), is obtained by combining Nash's idea ('The G Bound', Part II of [8]) with Aronson and Serrin's work ([1] and [2]). Set $u(s,y) = \Gamma_{a_1}(s, y; 0, x)$, in particular $\Gamma_a(1, x; 0, y) = \Gamma_{a_1}(s, y; 0, x)$ where $a_t = a(t - \cdot, \cdot)$. Consider

$$G(s) = \int e^{-\pi|y|^2} \log u(s,y)\, dy,$$

which is sensitive to areas where $|y|$ is not large and u is small. Note that $\int u(s,y)\, dy = 1$ and $G(s) < 0$. Roughly speaking, we obtain a result limiting the extent to which a fundamental solution can be very small over a large volume of space near its source point, y in $\Gamma(t,x;s,y)$. With additional restrictions on the solution that

$$\sup_{1/2\leq s\leq 1} u(x,y) \leq K$$

with an absolute constant K and the existence of R_λ depending only on λ satisfying

$$\sup_{1/2\leq s\leq 1}\int_{|y|>R_\lambda} u(s,y)\,dy \leq \frac{1}{2},$$

the lower bound for $G(1)$ is given that there is a constant $B<\infty$ depending only on λ such that for all $|x|\leq 1$

$$\int e^{-\pi|y|^2}\log\Gamma_a(1,x;0,y)\,dy \geq -B.$$

By taking $s=0$ and $t=2$, we write (Kolmogorov identity)

$$\Gamma_a(2,x;0,y) = \int \Gamma_a(1,\xi;0,y)\Gamma_{\tilde{a}}(1,x;0,\xi)\,d\xi$$

where $\tilde{a} = a(\cdot+1,\cdot)$. We overlap two fundamental solutions with nearby sources, and then there exists a constant C depending only on λ such that

$$\Gamma_a(t,x;s,y) \geq \frac{1}{C(t-s)^{n/2}}$$

for all x and y satisfying $|x-y|\leq\sqrt{t-s}$. The lower bound is obtained, for any $x,y\in\mathbb{R}^n$ and $t,s\in\mathbb{R}$, by repeatedly applying the previous techniques after taking subdivision of the region properly.

7.3 Consequences of the lower and upper bounds

From the lower and upper bounds (4), we derive Nash's theorem on the continuity of weak solutions and Moser's Harnack inequality. First, define

$$\operatorname{osc} u(s,\xi,R) = \sup\left\{|u(t,x)-u(t',x')| : s,s'\in B(\xi,R),\ s-R^2\leq t,\ t'\leq s\right\}.$$

Lemma 1. *For each $\delta \in (0,1)$, there is a $\rho = \rho(n,\lambda,\delta) \in (0,1)$ such that for all $(s,\xi) \in R \times R^n$ and $R > 0$:*

$$osc\ u(s,\xi,\delta R) \leq \rho\ osc\ u(s,\xi,R)$$

whenever $u \in C^\infty([s-R^2,s] \times \bar{B}(\xi,R))$ satisfies $Lu = 0$ in $(s-R^2,s) \times B(\xi,R)$.

Lemma 1 is a key to show following theorems, Theorems 2 and 3.

Theorem 2. *(NASH) For each $\delta \in (0,1)$, there are constants $C = C(n,\lambda,\delta) < \infty$ and $\beta = \beta(n,\lambda,\delta) \in (0,1)$ such that for all $(s,\xi) \in R \times R^n$ and $R > 0$:*

$$|u(t,x) - u(t',x')| \leq C\|u\|_{C_b([s-R^2,s]\times\bar{B}(\xi,R))}\left(\frac{|t-t'|^{1/2}+|x-x'|}{R}\right)^\beta$$

for $(t,x),(t',x') \in [s-(1-\delta^2)R^2,s] \times \bar{B}(\xi,(1-\delta)R)$ whenever $u \in C^\infty([s-R^2,s] \times \bar{B}(\xi,R))$ satisfies $Lu = 0$ in $(s-R^2,s) \times B(\xi,R))$

Theorem 3. *(Harnack inequality) Let $0 < \alpha < \beta < 1$ and $\gamma \in (0,1)$ be given. Then there is an $M = M(n,\lambda,\alpha,\beta,\gamma) < \infty$ such that for all $(s,x) \in R \times R^n$, all $R > 0$, and all non-negative $u \in C^\infty([s-R^2,s] \times \bar{B}(\xi,R))$ satisfying $Lu = 0$, one has*

$$u(t,y) \leq Mu(s,x) \tag{5}$$

for all $(t,y) \in [s-\beta R^2, s-\alpha R^2] \times \bar{B}(x,\delta R)$.

We also mention a Harnack inequality from Moser's paper (Theorem 1 on [7]) which is equivalent to Theorem 3.

Theorem 4. *If u is a non-negative weak solution satisfying $Lu = 0$ in $Q(t,x) = (t,t+\tau) \times B_{x,R}$ for some constants $\tau > 0$ and $R > 0$, then*

$$max_{(s,y)\in Q^-}\ u(s,y) \leq \gamma\ min_{(s,y)\in Q^+}\ u(s,y), \tag{6}$$

where $\gamma > 1$ is a constant which depends on n, λ and six geometrical constants $R, R', \tau, \tau_1^-, \tau_2^-, \tau^+$ and

$$Q^- = (\tau_1^-,\tau_2^-) \times B_{x,R'}, \quad Q^+ = (\tau^+,\tau) \times B_{x,R'}$$

with $0 < R' < R$ and $0 < \tau_1^- < \tau_2^- < \tau^+ < \tau$.

At the end of paper [8], Nash describes a Harnack inequality as

$$u(t,x_2) \geq F(u(t,x_1)/B, |x_1-x_2|/\sqrt{t-t_0}),$$

provided $0 \leq u \leq B$ for $t \geq t_0$. F is an a priori function that is a more general function than a linear function on u. From Theorem 3, it is not possible to obtain Nash's Harnack inequality.

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8 Two Weight Inequality for the Hilbert Transform

after Michael T. Lacey [1]
A summary written by Vjekoslav Kovač

Abstract

Consider all pairs of weights (w, σ) without common point masses. The problem is to characterize those pairs for which the Hilbert transform satisfies the estimate $\|\mathrm{H}(\sigma f)\|_{\mathrm{L}^2(w)} \leq C\|f\|_{\mathrm{L}^2(\sigma)}$. The paper in question contains a "real-variable characterization": the one in terms of the Poisson A_2 condition and the Sawyer-type testing conditions.

8.1 A bibliographical note

The study of Hilbert transform estimates with two different weights was first suggested by Benjamin Muckenhoupt and Richard L. Wheeden. The characterization was conjectured by Fedor Nazarov, Sergei Treil, and Alexander Volberg, who had also verified it in the case of doubling measures. The original paper by Michael T. Lacey claiming the result has been transformed into [2], a sequel to a previous paper [3]. Therefore, the complete proof of the two-weight characterization is distributed to the following pair of articles:

Part I, by Michael T. Lacey, Eric T. Sawyer, Chun-Yen Shen, and Ignacio Uriarte-Tuero, [3];

Part II, by Michael T. Lacey, [2].

A unified and self-contained exposition collecting material of both papers was given by Lacey [1]. Here we always refer to that paper.

8.2 Introduction and formulation of the main result

8.2.1 On individual two-weight problems

Let T be a given operator acting on one-dimensional functions. One can ask to determine for which weights w and σ the L^2 estimate

$$\|T(\sigma f)\|_{\mathrm{L}^2(w)} \leq \mathcal{N}\|f\|_{\mathrm{L}^2(\sigma)} \tag{1}$$

holds with a finite constant $\mathcal{N}$ independent of f. Here by a weight we simply mean a nonnegative locally finite Borel measure on $\mathbb{R}$, so the "weighted" L^2 norm is simply the Lebesgue norm with respect to that measure, i.e.

$$\|f\|_{\mathrm{L}^2(\sigma)} := \Big(\int_{\mathbb{R}} |f|^2 d\sigma \Big)^{1/2}.$$

The "product" σf is interpreted as the signed measure ν defined by $d\nu = f d\sigma$ and the expression $T(\sigma f)$ should make sense. One can further ask to quantify the dependence of the best possible constant $\mathcal{N}$ in (1) on the weights w, σ.

Let us quickly explain how Estimate (1) is motivated by the most classical weighted inequality for T, namely

$$\|Tf\|_{\mathrm{L}^2(w)} \leq \mathcal{N} \|f\|_{\mathrm{L}^2(w)}, \tag{2}$$

where this time w is a strictly positive locally integrable function on $\mathbb{R}$ and we can understand it as a Radon-Nikodym derivative of some positive measure. It is a classical idea of Eric T. Sawyer to write $\sigma = \frac{1}{w}$ and substitute $g = wf = \frac{f}{\sigma}$, so that

$$\|f\|^2_{\mathrm{L}^2(w)} = \int_{\mathbb{R}} |f(x)|^2 w(x) dx = \int_{\mathbb{R}} |g(x)|^2 \sigma(x) dx = \|g\|^2_{\mathrm{L}^2(\sigma)}$$

and (2) becomes (1) with f replaced with g. It is then natural to try to omit the pointwise constraint $w\sigma = 1$ and relax any conditions on w (such as the Muckenhoupt A_2 condition) to their joint variants in both w and σ. Finally, one does not have to confine themselves to absolutely continuous measures. We will also comment on the need for discussing general measures in the last section.

The *two-weight problem* was previously resolved for several classical operators:

- for the Hardy operator by Muckenhoupt,
- for the maximal function, the Poisson integral, and fractional integrals by Sawyer,
- for certain non-positive dyadic operators by Nazarov, Treil, and Volberg.

The discussed paper of Lacey establishes such a characterization for the Hilbert transform. This result is particularly interesting because the Hilbert transform is the first non-positive "continuous-type" operator for which this task is accomplished.

8.2.2 Characterization for the Hilbert transform

For any $0 < \tau < 1$ we consider the *truncated Hilbert transform* acting on a signed Borel measure ν by the formula

$$(\mathrm{H}_\tau \nu)(x) := \int_{\{y\,:\,\tau<|y-x|<\tau^{-1}\}} \frac{\nu(dy)}{y-x}. \tag{3}$$

One certainly recovers the usual definition of $\mathrm{H}_\tau f$ when ν is absolutely continuous and $d\nu = f d\lambda$, where λ denotes the Lebesgue measure. We never attempt to study the limiting behavior as $\tau \to 0$, because the limit of (3) need not exist in our setting, and because pointwise convergence is typically a more difficult problem than boundedness itself. We only take care that the inequality we investigate is uniform in τ, i.e. we are actually characterizing the bound

$$\sup_{0<\tau<1} \|\mathrm{H}_\tau(\sigma f)\|_{\mathrm{L}^2(w)} \le \mathcal{N} \|f\|_{\mathrm{L}^2(\sigma)}. \tag{4}$$

Let $\mathcal{N} \in [0,\infty]$ be the smallest constant such that (4) is satisfied. We suppress the dependence of H_τ on τ in the notation.

A convenient way of measuring size of the Poisson extension of σ to the upper half-plane is via the quantity

$$\mathrm{P}(\sigma, I) := \int_{\mathbb{R}} \frac{|I|}{\big(|I| + \mathrm{dist}(x, I)\big)^2}\, \sigma(dx),$$

defined for any bounded interval I. Define

$$\mathcal{A}_2 := \sup_I \mathrm{P}(\sigma, I)\mathrm{P}(w, I) \in [0,\infty], \tag{5}$$

where the supremum is taken over all bounded intervals I. Quantity (5) is called the *Poisson A_2 characteristic*. It is related to the Muckenhoupt A_2 characteristic

$$[w]_{\mathrm{A}_2} := \sup_I \Big(\frac{1}{|I|}\int_I \frac{1}{w}\Big)\Big(\frac{1}{|I|}\int_I w\Big)$$

by $[w]_{\mathrm{A}_2} \lesssim \mathcal{A}_2 \lesssim [w]_{\mathrm{A}_2}^2$ when w is a locally integrable function and $\sigma = \frac{1}{w}$. In general one only has the trivial estimate

$$\mathcal{A}_2 \ge \sup_I \frac{\sigma(I)w(I)}{|I|^2},$$

showing that the Poisson A_2 condition $\mathcal{A}_2 < \infty$ is indeed stronger than the two weight Muckenhoupt A_2 condition.

Let $\mathcal{T} \in [0, \infty]$ denote the smallest constant such that

$$\int_I \big(\mathrm{H}(\sigma \mathbf{1}_I)\big)^2 dw \leq \mathcal{T}^2 \sigma(I) \quad \text{and} \quad \int_I \big(\mathrm{H}(w \mathbf{1}_I)\big)^2 d\sigma \leq \mathcal{T}^2 w(I) \tag{6}$$

hold for all bounded intervals I. Conditions (6) are called the *Sawyer-type testing conditions*. It follows from Stefanie Petermichl's representation of H as an average of dyadic shifts that $\mathcal{T} \lesssim [w]_{\mathrm{A}_2}$ holds in the previously mentioned case $\sigma = \frac{1}{w}$.

Finally, let us say that measures σ and w *do not share a common point mass* if $\sigma(\{x\})w(\{x\}) = 0$ for each $x \in \mathbb{R}$. We can now formulate the main result of the paper in question.

Theorem 1 (M. T. Lacey, 2013)**.** *If two weights σ and w do not share a common point mass, then*

$$\mathcal{N} \simeq \mathcal{A}_2^{1/2} + \mathcal{T},$$

i.e. the two quantities are comparable.

8.2.3 On counterexamples

Even though it had initially been suspected that already the two weight Muckenhoupt A_2 condition might be sufficient for (4), it was soon shown by Muckenhoupt and Wheeden that this is not the case. Much later Nazarov gave a proof that the Poisson A_2 condition alone is not enough for having (4), answering negatively to the conjecture of Donald Sarason. A more advanced counterexample that also satisfies one of the two sets of testing conditions (6) was given by Nazarov and Volberg.

8.3 Proving the main theorem

A basic line of approach to this type of problems was invented by Nazarov, Treil, and Volberg. In order to keep this summary concise, we only comment on a couple of entirely novel ideas of Lacey, both of which can only be adapted to the case of the Hilbert transform. One can say (rather vaguely) that these novel ideas exploit the positivity, as the derivative of the Hilbert kernel is positive, $\frac{d}{dx}\frac{1}{y-x} = \frac{1}{(y-x)^2} > 0$.

For any bounded interval I let I_- and I_+ denote the left and right halves of I respectively. The associated Haar function is chosen depending on a weight σ and defined to be

$$\mathrm{h}_I^\sigma := \Big(\frac{\sigma(I_-)\sigma(I_+)}{\sigma(I)}\Big)^{1/2}\Big(\frac{\mathbf{1}_{I_+}}{\sigma(I_+)} - \frac{\mathbf{1}_{I_-}}{\sigma(I_-)}\Big).$$

This choice is convenient because h_I^σ is normalized in $\mathrm{L}^2(\sigma)$, its integral with respect to σ is 0, and its $\mathrm{L}^2(\sigma)$ inner product with $\iota\colon \mathbb{R} \to \mathbb{R}$, $\iota(x) := x$ is nonnegative.

8.3.1 The monotonicity principle

This ingredient turns certain "off-diagonal" estimates for H into estimates for the Poisson integrals.

Lemma 2. *If a positive measure μ and a signed measure ν are such that $|\nu| \le \mu$, both measures are supported outside an interval $I \in \mathcal{D}$, and none of μ, ν, w has a point mass at an endpoint of I, then*

$$\mathrm{P}(\mu, I)\Big\langle \frac{\iota}{|I|}, \overline{g}\Big\rangle_w \lesssim \langle \mathrm{H}\mu, \overline{g}\rangle_w$$

for any $g \in \mathrm{L}^2(J, w)$ such that $\int_J g dw = 0$.

Here

$$g \mapsto \overline{g} := \sum_{J' \in \mathcal{D}} |\langle g, \mathrm{h}_{J'}^w\rangle_w| \ \mathrm{h}_{J'}^w$$

is a Haar multiplier with respect to a dyadic grid in question.

8.3.2 The energy inequality

The following auxiliary result fundamentally uses both the $\mathcal{A}_2$ condition and the testing conditions. The energy of w with respect to an interval I from a dyadic grid $\mathcal{D}$ is defined to be

$$\mathrm{E}(w, I) := \Big(|I|^{-2} \sum_{J \in \mathcal{D} : J \subseteq I} \langle \iota, \mathrm{h}_J^w\rangle_w^2\Big)^{1/2}.$$

Lemma 3. *For an interval $I_0 \in \mathcal{D}$ and its partition $\mathcal{P}$ into intervals such that neither σ nor w have point masses at their endpoints one has the estimate*

$$\sum_{I \in \mathcal{P}} \mathrm{P}(\sigma, I)^2 \mathrm{E}(w, I)^2 w(I) \lesssim (\mathcal{A}_2 + \mathcal{T}^2)\, \sigma(I_0).$$

There is also a more advanced *functional energy inequality*, which also crucially depends on positivity.

8.4 Applications and connections with other problems

The two weight problem for the Hilbert transform is connected with a number of diverse problems, such as composition of Toeplitz operators, model spaces, and de Branges spaces. We only shortly comment on the first topic.

Let $\mathrm{H}^2(\mathbb{D})$ denote the complex Hardy space on the unit disk and let $\mathrm{P}_+ \colon \mathrm{L}^2(\mathbb{T}) \to \mathrm{H}^2(\mathbb{D})$ denote the Riesz projection. A *Toeplitz operator* with symbol $g \in \mathrm{L}^2(\mathbb{T})$ is a map T_g defined by

$$T_g f := \mathrm{P}_+(gf).$$

Finally, let

$$\mathrm{P}(|g|^2)(z) := \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - z\bar{\theta}|^2} |g(\theta)|^2 d\theta \quad \text{for } z \in \mathbb{D}$$

denote the Poisson extension of $|g|^2$ to the unit disk.

Conjecture 4 (D. Sarason, 1994)**.** *For two outer functions $g, h \in \mathrm{H}^2(\mathbb{D})$ the composition $T_g T_{\overline{h}}$ is bounded on $\mathrm{H}^2(\mathbb{D})$ if and only if*

$$\sup_{z \in \mathbb{D}} \mathrm{P}(|g|^2)(z) \mathrm{P}(|h|^2)(z) < \infty. \tag{7}$$

It was observed by Treil that (7) is a necessary condition for boundedness, which motivated the conjecture. However, the previously mentioned result by Nazarov shows that (7) is not sufficient, so Sarason's conjecture is actually false.

However, the problem of characterizing all pairs (g, h) for which $T_g T_{\overline{h}}$ is bounded on $\mathrm{H}^2(\mathbb{D})$ remains interesting. It is equivalent to boundedness of $\mathrm{M}_g \mathrm{P}_+ \mathrm{M}_{\overline{h}}$, where M_g denotes the multiplication by g. Using the structure of outer functions one can further reformulate it as boundedness of $\mathrm{M}_{|g|}\mathrm{P}_+\mathrm{M}_{|h|}$ on $\mathrm{L}^2(\mathbb{T})$. Since

$$\big\|\mathrm{M}_{|g|}\mathrm{P}_+\mathrm{M}_{|h|}\big\|_{\mathrm{L}^2\to\mathrm{L}^2} = \big\|\mathrm{P}_+(|h|\cdot)\big\|_{\mathrm{L}^2\to\mathrm{L}^2(|g|^2d\lambda)} = \big\|\mathrm{P}_+(|h|^2\cdot)\big\|_{\mathrm{L}^2(|h|^2d\lambda)\to\mathrm{L}^2(|g|^2d\lambda)}$$

we arrive precisely at the two weight inequality for P_+, with weights $|g|^2d\lambda$ and $|h|^2d\lambda$. By writing $\mathrm{P}_+ = I - \frac{\pi}{i}\mathrm{H}$ it becomes possible to apply the characterization of Theorem 1.

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9 L^p theory for outer measures and two themes of Lennart Carleson united

after Y. Do and C. Thiele [1]
A summary written by Mariusz Mirek

Abstract

We will develop a theory of L^p spaces based on outer measures instead of measures. With the aid of this theory we illustrate L^p boundedness of bilinear Hilbert transform.

9.1 Introduction

The time–frequency analysis provides some tools/techniques which allows us to bound various operators (multilinear operators) corresponding with model sums of the form

$$\Lambda(f_1, \dots, f_n) = \sum_{P \in \mathbf{P}} c_P \prod_{j=1}^{n} a_P(f_j),$$

where the summation index runs through a discrete set $\mathbf{P}$, typically a collection of rectangles (tiles) in the phase plane and the coefficients c_P are inherent to the multilinear form. One of the most important example of the sequences $(a_P(f_j))_{P \in \mathbf{P}}$ is

$$a_P(f_j) = \langle f_j, \phi_P \rangle,$$

where

$$\phi_P(x) = 2^{-k}\phi(2^{-k}x - n)e^{2\pi i 2^{-k}xl},$$

is L^1 normalized wave packet with a suitably chosen Schwartz function ϕ and integers k, l, n which parameterize the space $\mathbf{P}$. For more examples of sequences $(a_P(f_j))_{P \in \mathbf{P}}$ we refer to [3].

The main idea of Yen Do and Christoph Thiele in [1] is based on a brilliant observation that the bound on Λ is a Hölder's inequality with respect to an outer measure on the space $\mathbf{P}$

$$|\Lambda(f_1, \dots, f_n)| \leq C \prod_{j=1}^{n} \|a_P(f_j)\|_{L_j^{p_j}(\mathbf{P}, \dots)},$$

where ... stands for the explicit outer measure structure which will be discussed later. Our principal examples will be paraproducts and the bilinear Hilbert transform. However, after the general use of Hölder's inequality the main work is carried over into the following estimate

$$\|a_P(f_j)\|_{L_j^{p_j}(\mathbf{P},\dots)} \leq C\|f_j\|_{L^{p_j}(\mathbb{R})},$$

for each $1 \leq j \leq n$ separately. Such kind of estimate will be called a generalized Carleson embedding theorem. Employing these ideas we will reprove the following theorem of Michael Lacey and Christoph Thiele [2].

Theorem 1. *Let $\beta = (\beta_1, \beta_2, \beta_3)$ be a vector in $\mathbb{R}^3$ with pairwise distinct entries. For three Schwartz functions f_1, f_2, f_3 on the real line we define*

$$\Lambda_\beta(f_1, f_2, f_3) = p.v. \int_{\mathbb{R}} \int_{\mathbb{R}} \prod_{j=1}^{3} f_j(x - \beta_j t) \frac{dt}{t} dx.$$

Then for any $2 < p_1, p_2, p_3 < \infty$ with $\sum_{j=1}^{3} \frac{1}{p_j} = 1$ there is a finite constant $C > 0$ such that for all Schwartz functions f_1, f_2, f_3 we have

$$|\Lambda_\beta(f_1, f_2, f_3)| \leq C \prod_{j=1}^{3} \|f_j\|_{L^{p_j}(\mathbb{R})}.$$

Before we prove this theorem let us recall that outer measures are subadditive set function. Some outer measures produce interesting measures by restriction to Caratheodory measurable sets, the best known example is classical Lebesgue theory. The outer measures which will occur in time–frequency analysis give rise only to trivial Caratheodory measurable sets.

Outer measures do not satisfy additivity for disjoint collections of sets and we do not have a linear theory of integrals with respect to outer measures. Even though, we are led to a non–linear theory and instead of linear functionals we have to consider norms, the L^p theory of outer measure spaces is quite parallel to the standard theory of L^p spaces.

9.2 L^p theory of outer measure spaces

In many cases it is convenient to generate an outer measure by a concrete premeasure. It is possible due to the following.

Proposition 2. *Let X be a set and $\mathbf{E}$ be a collection of subsets of X. Let σ be a function from $\mathbf{E}$ to $[0, \infty)$. Define for an arbitrary subset E of X*

$$\mu(E) = \inf_{\mathbf{E}'} \sum_{E' \in \mathbf{E}'} \sigma(E'),$$

where the infimum is taken over all subcollections $\mathbf{E}'$ of $\mathbf{E}$ which cover E. Then μ is an outer measure. Moreover, if for every set $E \in \mathbf{E}$ and every cover of E by a subcollection $\mathbf{E}'$ of $\mathbf{E}$ we have

$$\sigma(E) \leq \sum_{E' \in \mathbf{E}'} \sigma(E'),$$

then $\mu(E) = \sigma(E)$ for every $E \in \mathbf{E}$.

9.2.1 Examples of outer measures

Example 1. *Lebesgue measure via dyadic cubes.* Let X be the Euclidean space $\mathbb{R}^d$ for some $d \geq 1$ and let $\mathbf{E}$ be the set of all dyadic cubes Q of the form

$$Q = [2^k n_1, 2^k(n_1 + 1)) \times \ldots \times [2^k n_d, 2^k(n_d + 1)),$$

where $k, n_1, \ldots, n_d \in \mathbb{Z}$. For each dyadic cube Q we set $\sigma(Q) = 2^{kd}$. Then σ generates an outer measure μ which is the classical outer measure on $\mathbb{R}^d$ and we have $\sigma(Q) = \mu(Q)$ for any $Q \in \mathbf{E}$.

Example 2. *Lebesgue measure via balls.* Let $X = \mathbb{R}^d$ be as above and $\mathbf{E}$ be the set of all open balls $B_r(x)$ with radius $r > 0$ and canter $x \in \mathbb{Q}^d$. Let $\sigma(B_r(x)) = r^d$ for any $B_r(x) \in \mathbf{E}$. Then σ generates a multiple of Lebesgue outer measure μ and again $\sigma = \mu_{|\mathbf{E}}$.

Example 3. *Outer measure generated by tents.* Let $X = \mathbb{R} \times (0, \infty)$ be the open upper half plane and let $\mathbf{E}$ be the set of all open isosceles triangles of the form

$$T(x, s) = \{(y, t) \in \mathbb{R} \times (0, \infty) : t < s,\ |x - y| < s - t\},$$

for some pair $(x, s) \in \mathbb{R} \times (0, \infty)$ which is the tip of the tent. Define $\sigma(E) = s$ for any $T(x, s) \in \mathbf{E}$. One can easily see that σ generates an outer measure μ on X which coincides with σ on the collection $\mathbf{E}$. Moreover, the Caratheodory's σ–algebra $\mathfrak{M}$ is trivial, i.e. $\mathfrak{M} = \{\emptyset, X\}$.

9.2.2 Size, essential supremum and super level measure

To avoid too abstract settings we shall assume that X is a metric space and every set of the collection $\mathbf{E}$ is Borel. Finally $\mathcal{B}(X)$ will denote the set of all Borel measurable functions on X.

Definition 3. *Let X be a metric space. Let σ be a function on a collection $\mathbf{E}$ of Borel subsets of X and let μ be the outer measure generated by σ such that $\sigma = \mu_{|\mathbf{E}}$. A size map is a map $S : \mathcal{B}(X) \mapsto [0, \infty]^{\mathbf{E}}$ satisfying for every $f, g \in \mathcal{B}(X)$ and every $E \in \mathbf{E}$ the following properties:*

1. *Monotonicity: if $|f| \leq |g|$, then $S(f)(E) \leq S(g)(E)$.*
2. *Scaling: $S(\lambda f)(E) = |\lambda| S(f)(E)$, for every $\lambda \in \mathbb{C}$.*
3. *Quasi–subadditivity: $S(f + g)(E) \leq C(S(f)(E) + S(g)(E))$, for some finite constant $C > 0$ depending only on S but not on f, g, E.*

Now we discuss sizes for the examples from previous subsection. In the Example 1, we define for every $f \in \mathcal{B}(X)$ and every cube $Q \in \mathbf{E}$

$$S(f)(Q) = \mu(Q)^{-1} \int_Q |f(x)| dx,$$

where the integral is in the Lebesgue sense. In the Example 2 we may define size in a similar way as above taking open balls instead of cubes.

In the Example 3 for every $F \in \mathcal{B}(\mathbb{R} \times (0, \infty))$ and every tent $T(x, s) \in \mathbf{E}$ we define

$$S(F)(T(x, s)) = \frac{1}{s} \int_{T(x,s)} |F(y, t)| dy \frac{dt}{t}.$$

In the literature one often works with the class of Borel measures ν on X rather than the class of Borel measurable functions and defines

$$S(\nu)(T(x, s)) = s^{-1} |\nu|(T(x, s)).$$

If $S(\nu)$ is bounded, the measure ν is called a Carleson measure on X.

Finally, we introduce the definition of outer essential supremum and super level measure which allow us to build up theory of outer L^p spaces in the next subsection.

Definition 4. *Assume that* (X, μ, S) *is an outer measure space with a size map* S*. Given a Borel subset* F *of* X *we define the outer essential supremum of* $f \in \mathcal{B}(X)$ *on* F *to be*

$$\text{outsup}_F S(f) = \sup_{E \in \mathbf{E}} S(F\mathbf{1}_F)(E).$$

The use of the outer essential supremum is the main subtle point in the following.

Definition 5. *Assume that* (X, μ, S) *is an outer measure space with a size map* S*. Let* $f \in \mathcal{B}(X)$ *and* $\lambda > 0$ *and define the super level measure to be*

$$\mu(S(f) > \lambda) = \inf\{\mu(F) : \ F \in \mathbf{E} \quad \textit{and} \quad \text{outsup}_{F^c} S(f) \leq \lambda\}.$$

We emphasize that in general $\mu(S(f) > \lambda)$ is not the outer measure of the Borel set where $|f|$ is larger that λ.

9.2.3 Outer L^p spaces

The definition of outer L^p space is straightforward using outer essential supremum and super level measure. We will follow in the same way as in the classical theory.

Definition 6. *Let* (X, μ, S) *be an outer measure space with a size map* S*. Then for any* $0 < p \leq \infty$ *we define the spaces*

$$L^\infty(X, \mu, S) = \{f \in \mathcal{B}(X) : \ \|f\|_{L^\infty(X,\mu,S)} = \text{outsup}_X S(f) < \infty\},$$

$$L^p(X, \mu, S) = \{f \in \mathcal{B}(X) : \ \|f\|_{L^p(X,\mu,S)} = \left(\int_0^\infty p\lambda^{p-1} \mu(S(f) > \lambda) d\lambda \right)^{1/p} < \infty\},$$

$$L^{p,\infty}(X, \mu, S) = \{f \in \mathcal{B}(X) : \ \|f\|_{L^{p,\infty}(X,\mu,S)} = \left(\sup_{\lambda > 0} \lambda^p \mu(S(f) > \lambda) \right)^{1/p} < \infty\}.$$

Proposition 7. *Assume that* (X, μ, S) *is an outer measure space with a size map* S *and let* $f, g \in \mathcal{B}(X)$*. Then for every* $0 < p \leq \infty$ *we have*

1. *Monotonicity: if* $|f| \leq |g|$*, then* $\|f\|_{L^p(X,\mu,S)} \leq \|g\|_{L^p(X,\mu,S)}$*.*
2. *Scaling:* $\|\lambda f\|_{L^p(X,\mu,S)} = |\lambda| \|f\|_{L^p(X,\mu,S)}$*, for every* $\lambda \in \mathbb{C}$*.*

3. *Quasi–subadditivity:* $\|f+g\|_{L^p(X,\mu,S)} \le C(\|f\|_{L^p(X,\mu,S)} + \|g\|_{L^p(X,\mu,S)})$, *for some finite constant* $C>0$ *depending only on* S *but not on* f, g, E.

The same type of estimates hold for the spaces $L^{p,\infty}(X,\mu,S)$. *Moreover,* $\|f\|_{L^{p,\infty}(X,\mu,S)} \le \|f\|_{L^p(X,\mu,S)}$ *for any* $f \in L^p(X,\mu,S)$.

Now we have the following generalization of Hölder's inequality.

Proposition 8. *Assume that* (X,μ,S) *is an outer measure space with a size map* S *and there are two sizes* S_1, S_2 *such that for any* $E \in \mathbf{E}$ *and* $f_1, f_2 \in \mathcal{B}(X)$ *we have*

$$S(f_1 f_2)(E) \le S_1(f_1)(E) S_2(f_2)(E).$$

Then for any $p, p_1, p_2 \in (0,\infty]$ *such that* $1/p = 1/p_1 + 1/p_2$ *we have*

$$\|f_1 f_2\|_{L^p(X,\mu,S)} \le 2\|f_1\|_{L^{p_1}(X,\mu,S)} \|f_2\|_{L^{p_2}(X,\mu,S)}.$$

9.2.4 Paraproducts and Carleson embedding theorem

Now we are at the place where we can show how this theory works in practise. Let $X = \mathbb{R} \times (0,\infty)$ be the upper half plane and $\mathbf{E}$ be the collection of tents as in the Example 3. For $1 \le p < \infty$, define sizes

$$S_p(F)(T(x,s)) = \left(\frac{1}{s} \int_{T(x,s)} |F(y,t)|^p dy \frac{dt}{t} \right)^{1/p},$$

and $S_\infty(F)(T(x,s)) = \sup_{(y,t)\in T(x,s)} |F(y,t)|$, where $F \in \mathcal{B}(\mathbb{R} \times (0,\infty))$.

For $f \in L^\infty(\mathbb{R})$ consider the function $F_\phi(f)$ on X defined as follows

$$F_\phi(f)(y,t) = \int_X f(z) t^{-1} \phi(t^{-1}(y-z)) dz,$$

where ϕ is a Schwartz function such that $\int_{\mathbb{R}} f(x)dx = 0$. It is well known that $|F_\phi(f)(y,t)|^2$ is a Carleson measure. The mapping $f \mapsto F_\phi(f)$ is an embedding of a space of functions on the real line into a space of functions in the upper half plane reminiscent of Carleson embeddings. Therefore, Carleson embedding theorem can be rephrased in the following way.

Proposition 9. *Let $1 < p \leq \infty$ and ϕ be a Schwartz function as above. Then*

$$\|F_\phi(f)\|_{L^p(X,\mu,S_\infty)} \leq C_{\phi,p}\|f\|_{L^p(\mathbb{R})},$$

and in addition if $\int_{\mathbb{R}} \phi(x)dx = 0$, then

$$\|F_\phi(f)\|_{L^p(X,\mu,S_2)} \leq C_{\phi,p}\|f\|_{L^p(\mathbb{R})}.$$

This proposition will be the main tool in the estimates for paraproducts. A classical paraproduct is a bilinear operator, which after pairing with a third function becomes a trilinear form which is essentially of the type

$$\Lambda(f_1, f_2, f_3) = \int_0^\infty \int_{\mathbb{R}} \prod_{j=1}^{3} F_{\phi_j}(f_j)(x,t)dx\frac{dt}{t},$$

where ϕ_1, ϕ_2, ϕ_3 are Schwartz functions such that $\int_{\mathbb{R}} \phi_k(x)dx = 0$, for $k = 1, 2$, whereas the third one does not necessarily have mean zero. Now one can see that Hölder's inequality combined with Carleson emmbeding theorem yield that

$$\begin{aligned}|\Lambda(f_1, f_2, f_3)| &\leq C\Big\| \prod_{j=1}^{3} F_{\phi_j}(f_j)\Big\|_{L^1(X,\mu,S_1)} \\ &\leq C\|F_{\phi_1}(f_1)\|_{L^{p_1}(X,\mu,S_2)}\|F_{\phi_2}(f_2)\|_{L^{p_2}(X,\mu,S_2)}\|F_{\phi_3}(f_3)\|_{L^{p_3}(X,\mu,S_\infty)} \\ &\leq C\|f_1\|_{L^{p_1}(\mathbb{R})}\|f_2\|_{L^{p_2}(\mathbb{R})}\|f_3\|_{L^{p_3}(\mathbb{R})},\end{aligned}$$

for any $1 < p_1, p_2, p_3 < \infty$ such that $\sum_{j=1}^{3} 1/p_j = 1$.

9.2.5 Boundedness of the bilinear Hilbert transform

Exploring the same ideas as for Λ from the previous subsection we will be able to prove similar estimates for Λ_β from Theorem 1. For this purpose we have to consider the space $X = \mathbb{R} \times \mathbb{R} \times (0, \infty)$ and define generalized tents

$$T^B(x, \xi, s) = \{(y, \eta, t) \in X : t \leq s,\ |x - y| \leq s - t,\ |\xi - \eta| \leq Bt^{-1}\},$$

for $B > 0$. Let $0 < b < 1 < B$ and for $F \in \mathcal{B}(X)$ define the size map by setting

$$S^{b,B}(F)(T(x, \xi, s)) =$$

$$\max\left\{\left(\int_{T^B(x,\xi,s)\setminus T^b(x,\xi,s)} |F(y, \eta, t)|^2 dyd\eta dt\right)^{1/2}, \sup_{(y,\eta,t)\in T^B(x,\xi,s)} |F(y, \eta, t)|\right\},$$

and $\sigma(S^{b,B}(F)(T(x,\xi,s))) = s$. The main ingredient in the proof of Theorem 1 and the novelty in the time–frequency analysis is the following generalized Carleson theorem.

Theorem 10. *Let $B \geq 10b$, $2 \leq p \leq \infty$ and define for $f \in L^p(\mathbb{R})$ the function*

$$F(y,\eta,t) = \sup_{\phi\in\Phi}\left|\int_{\mathbb{R}} f(x)e^{i\eta(y-x)}t^{-1}\phi(t^{-1}(y-x))dx\right|,$$

where the supremum is taken over the class Φ of all functions $\phi \in L^2(\mathbb{R})$ such that $\widehat{\phi}$ is supported in $(-b/2,b/2)$ and $|\phi(x)| \leq 1/b(1+|x/b|)^{-3}$ and $|\phi'(x)| \leq 1/b(1+|x/b|)^{-2}$ for all $x \in \mathbb{R}$. Then for any $2 < p \leq \infty$ we have

$$\|F_\phi(f)\|_{L^p(X,\mu,S_\infty)} \leq C_{p,b,B}\|f\|_{L^p(\mathbb{R})},$$

and

$$\|F_\phi(f)\|_{L^{2,\infty}(X,\mu,S_\infty)} \leq C_{b,B}\|f\|_{L^2(\mathbb{R})}.$$

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10 Rectifiable sets and the Traveling Salesman Problem

after P. Jones [3]

A summary written by Martí Prats

Abstract

We present the so-called Peter Jones betas and we use them to give a necessary and sufficient condition for a set to be rectifiable.

10.1 Introduction

A salesman wants to visit a number of villages during a certain period and then go back home. Which is the best order to spend the minimal time traveling? Of course we seek the shortest cycle through all this places. If we represent each village with a point in the plane, let's call this set E, we can use the so-called greedy algorithm to find a spanning tree G of minimal length with vertices in all the points in E, and one such minimal tree will be contained in any minimal tour T with segment end-points in the points in E. A minimal connected set K connecting all those villages may not have segment endpoints in the points of E, but

$$\ell(K) \leq \ell(G) < \ell(T) \leq 2\ell(K)$$

and, thus, the minimal connected set bounds the distance that a clever salesman must drive (as long as he can travel in helicopter).

When it comes to a non-finite set E, the Traveling Salesman Problem consists in finding a minimal rectifiable curve $\Gamma \supset E$. This will be possible only when the set is rectifiable, that is, when the set is contained in the image of a finite interval by a Lipschitz function. One necessary condition for E to be rectifiable is that the Hausdorff one-dimensional (outer) measure of the set, $\mathcal{H}^1(E)$, is finite, but it is not sufficient unless E is connected.

Let Δ_k be the canonical dyadic grid of cubes which have side-length 2^{-k} and $\Delta = \bigcup_k \Delta_k$ the whole dyadic mesh. We will give a characterization of rectifiable sets in terms of the next coefficients:

Definition 1. *Let Q be a (dyadic) square of side $\ell(Q)$. We write $3Q$ for the concentric square with triple side-length, and call*

$$\beta_E(Q) = \frac{2}{\ell(3Q)} \inf\{\sup_{E\cap 3Q} \operatorname{dist}(z, L) : \ L \textit{ is any line}\}.$$

Then $\beta_E(Q)\ell(3Q)$ is the width of the narrowest strip containing $E \cap 3Q$. Notice that $\beta_E(Q) < 1$.

Definition 2. *Given a set E, we associate to it the coefficient*

$$\beta^2(E) = \operatorname{diam}(E) + \sum_{\substack{Q \in \Delta \\ \ell(Q) \leq \operatorname{diam}(E)}} \beta_E^2(Q)\ell(Q).$$

The main result of this paper is the next theorem:

Theorem 3. *Suppose $E \subset \mathbb{C}$ is a bounded set. Then E is contained in a rectifiable curve if and only if $\beta^2(E)$ is finite. Moreover, there are constants c_1, c_2 such that*

$$c_1\beta^2(E) \leq \inf_{\Gamma \supset E} \mathcal{H}^1(\Gamma) \leq c_2\beta^2(E). \tag{1}$$

Notice that, even though we do not find the best path for the salesman, we bound the distance the salesman must travel if he designs his route wisely.

We follow the proof of the book by Garnett and Marshall [2], which is mostly inspired by the original Peter Jones' paper [3]. The argument of the left-hand side inequality in (1) is mainly of complex analysis. In [4], Kate Okikiolu gives a more general proof valid in $\mathbb{R}^n$.

10.2 Right-hand inequality: finding a good route

Sketch of the proof. **Breaking a rectangle into two**
We start giving a construction to iterate. Consider a given rectangle S with side-lengths L and βL with $\beta < 1$ whose four sides intersect E. We find two new rectangles S_0 and S_1 with long sides L_0 and L_1 with minimal width which contain $E \cap S$ and separate the left and right thirds of S. If the middle third of the rectangle is empty, we add a segment T. Otherwise, we ensure the new rectangles have at least one common point and it is in E.

One can see using the Pythagorean Theorem that in both cases we have

$$L_0 + L_1 \leq (1 + 5\beta^2)L \tag{2}$$

and

$$|T| \leq (1 + \beta^2)L \tag{3}$$

Covering E
Assume $E \subset Q_0 \in \Delta_0$ and let S_0 be a rectangle containing E, meeting E

at its four sides and with minimal shortest side. Write its side-lengths as L_0 and βL_0. Notice that $\beta \leq C\beta_E(Q_0)$. Iterate the process described above.

After n steps we have 2^n rectangles S_I, with $I \in \{0,1\}^n$ and, if case 2 is applied to S_I, we get a segment T_I. One can see that, after 25 steps the diameter of S_I will drop by at least 1/2.

Bounds for the length

Let R_n be the sum of the diameters of the rectangles at stage n. From (2) we have

$$L_0 + L_1 \leq L + C\beta_E^2(Q)\ell(Q) \tag{4}$$

for any Q such that $Q \cap E \cap S \neq \emptyset$ and $\mathrm{diam}(S) \leq \ell(Q) < 2\mathrm{diam}(S)$. As noted before, each such cube can occur at most a certain number of times during iterations, so that $R_n \leq C\beta^2(E)$.

It only remains to bound the sum of the lengths of all the segments. Note that, if case 2 is applied and $L_0 + L_1 \geq 0.9L$, then the original rectangle couldn't be flat, so $\beta \geq \beta_0$ for a fixed β_0, and so, using (3) we get

$$|T| \leq C\beta_E^2(Q)\ell(Q).$$

Thus, the sum of lengths of all the segments created from a rectangle with $\beta \geq \beta_0$ is at most $C\beta^2(E)$.

Now write $R_n = I_n + II_n$ where I_n is the sum of the lengths of the rectangles at stage n to which case 1 will be applied or for which $\beta \geq \beta_0$ and II_n the sum of the lengths of the remaining rectangles. Notice that in the second case, $L_0 + L_1 < 0.9L$. Let T_{n+1} be the sum of the segments created at stage n with $\beta < \beta_0$. Then, it can be seen using the previous bounds that

$$\sum_{j=1}^{n} 0.1T_j \leq R_n - R_0 + C \sum_{Q \in \Delta : Q \subset Q_0} \beta_E^2(Q)\ell(Q) \leq C\beta^2(E).$$

Glue rectangles of stage n with all the segments created and take limit. $\square$

10.3 The route cannot be improved much

10.3.1 The left-hand inequality for Lipschitz graphs

Let Γ be the graph of a Lipschitz function. Then (1) holds for all $E \subset \Gamma$.

Sketch of the proof. We assume that $\Gamma = \{0 \leq x \leq 1, y = f(x)\}$, where $f(0) = f(1)$ and f is Lipschitz with constant M.

It is enough to show the case $E = \Gamma$. Let I_j^n be the jth dyadic interval of length 2^{-n}, call its image graph Γ_j^n and let J_j^n be the segment uniting the endpoints of Γ_j^n. Then J_{2j}^{n+1}, J_{2j+1}^{n+1} and J_j^n are the three sides of a triangle and call $\delta_{n,j}$ its height times 2^n. Using the Pythagorean Theorem, one gets

$$2^{-n}\delta_{n,j}^2 \lesssim \ell(J_{2j}^{n+1}) + \ell(J_{2j+1}^{n+1}) - \ell(J_j^n)$$

with constant depending on the Lipschitz constant M. This implies

$$\sum_{m,k} c2^{-m}\delta_{m,k}^2 \leq 2\ell(\Gamma).$$

Now, by the triangular inequality,

$$\beta_{n,j} := 2^n \sup\{\mathrm{dist}(z, J_j^n) : z \in \Gamma_j^n\} \leq \sum_{m=n}^{\infty} 2^{n-m} \sup\{\delta_{m,k} : I_k^m \subset I_j^n\}.$$

Using Hölder inequalities and other standard arguments for series, one gets

$$\sum_{n,j} 2^{-n}\beta_{n,j} \lesssim \sum_{m,k} 2^{-m}\delta_{m,k}^2 \lesssim 2\ell(\Gamma).$$

Finally we extend the function periodically and obtain some translated coefficients $\beta_{n,j}(t)$ related to $\Gamma(t) = (Id \times f)([t, 1+t]) \subset \mathbb{C}$ verifying the last inequality as well. Then, for a cube Q with $\ell(Q) = 2^{-n-2}$, $3Q$ will have projection contained in the translation of an interval $I_{n,j}(t)$ with probability $1/4$ with respect to the Lebesgue measure on t, so

$$\sum_{\ell(Q)=2^{-n-2}} \beta_\Gamma^2(Q) \lesssim \int_{-1}^{1} \sum_j \beta_{n,j}(t)^2 dt.$$

Summing with respect to n proofs the claim for Lipschitz periodic functions. □

10.3.2 The general case: a decomposition theorem

We call an M-Lipschitz domain to a simply connected domain whose boundary can be expressed as $\{r(\theta)e^{i\theta} : 0 \leq \theta < 2\pi\}$ (i.e. it is starlike with respect to the origin), with r a Lipschitz function of coefficient M and $\frac{1}{M+1} \leq r(\theta) \leq 1$.

In the previous section we have established (1) for boundaries of M-Lipschitz domains, so we need to extend the result to general sets E of finite length. The key point is to find $E \subset \bigcup \Gamma_j$ being each Γ_j the boundary of an M-Lipschitz domain $\mathcal{D}_j$. We need to do this in such a way that we keep control on the total length and on the relations between the original betas and $\sum_Q \sum_{\Gamma_j} \beta^2_{\Gamma_j}(Q)$. The latter is a rather technical lemma we skip here for the sake of brevity. Using this lemma relating the betas, Theorem 3 gets reduced to the next one:

Theorem 4. *There exists a constant $M < \infty$ such that if Γ is a connected plane set with $\mathcal{H}^1(\Gamma) < \infty$, then there exists a connected plane set $\widetilde{\Gamma} \supset \Gamma$ such that $\mathcal{H}^1(\widetilde{\Gamma}) \le M\mathcal{H}^1(\Gamma)$, the bounded components $\mathcal{D}_j$ of $\mathbb{C} \setminus \widetilde{\Gamma}$ are M-Lipschitz domains with $\Gamma \subset \bigcup \partial\mathcal{D}_j$, and the boundary of the unbounded component $\mathcal{D}_0$ of $\mathbb{C} \setminus \widetilde{\Gamma}$ is a circle at least $3\sqrt{2}\mathcal{H}^1(\Gamma)$ units from Γ.*

Outline of the proof. We may proof the case of Γ being the boundary of a simply connected domain Ω. Otherwise, we could apply this particular result to each bounded component of the original set united to a circle big enough by a segment.

We will apply a corona construction to a disk, related to Ω through a Riemann mapping φ. We will make the division in the disk in such a way that, using the properties of φ we can ensure that the images of the domains in $\mathbb{D}$ are also M-Lipschitz domains.

Write $F = \sqrt{\varphi'}$ and $g = \log(\varphi')$. Using Bieberbach's Theorem one can see that g is in the Bloch space with seminorm

$$\|g\|_{\mathcal{B}} \le 6 \tag{5}$$

i.e. $\frac{\varphi''(z)}{\varphi'(z)} \le \frac{6}{1-|z|^2}$ for all $z \in \mathbb{D}$.

Thanks to a result due to Alexander (see [1]) we can see that $\varphi' \in H^1$, and $\|\varphi'\|_{H^1} \le 2\mathcal{H}^1(\partial\Omega)$. With this result and the Littlewood-Paley formula for Hardy spaces, one can see that

$$\int\int_{\mathbb{D}} |\varphi'(z)||g'(z)|^2 \log\frac{1}{|z|} dm(z) \le 4\mathcal{H}^1(\partial\Omega). \tag{6}$$

Set $\mathcal{D}_0 = \{|z| \le 1/2\}$ and $\mathcal{U}_0 = \varphi(\mathcal{D}_0)$. By the growth theorem and the distortion theorem for univalent functions (see [2, Theorem I.4.5]), one can see that $\mathcal{U}_0$ is an M-Lipschitz domain. Since $\varphi \in H^1$ we also have

$$\mathcal{H}^1(\partial\mathcal{U}_0) \le \mathcal{H}^1(\partial\Omega). \tag{7}$$

Next form the dyadic Carleson boxes

$$Q = \left\{re^{i\theta} : 1 - 2^{-n} \leq r < 1, \pi j 2^{-(n+1)} \leq \theta < \pi(j+1)2^{-(n+1)}\right\}$$

for $0 \leq j < 2^{n+2}$, and consider their top halves $T(Q) = \{z \in Q : |z| < 1 - 2^{-(n+1)}\}$. Write z_Q for the center of $T(Q)$. We choose the domains by a stoping time argument. Fix ε to be determined later and consider a Carleson box Q as big as possible.

Define $G(Q)$ to be the set of maximal boxes $Q' \subset Q$ for which

$$\sup_{T(Q')} |g(z) - g(z_Q)| \geq \varepsilon \tag{8}$$

and define $\mathcal{D}(Q) = \left(Q \setminus \bigcup_{G(Q)} Q'\right) \cup T(Q)$.

Keep finding $\mathcal{D}(Q)$ for the successive remaining maximal cubes. Then, the family $\{\mathcal{D}_j\}_{j\geq 0}$ is pairwise disjoint. Write $\mathcal{U}_j = \varphi(\mathcal{D}_j)$.

To bound the lengths one must distinguish three types of domains and use different techniques for each one of them. We refer the reader to [2] or [3] for the details. We obtain

$$\sum \mathcal{H}^1(\partial \mathcal{U}_j) \leq C\mathcal{H}^1(\partial\Omega).$$

The last step is to divide the original domains into smaller ones which are M-Lipschitz. Here one uses a Cantor-type construction and uses again complex analysis techniques to keep the previous bound. □

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11 The Cauchy integral, analytic capacity, and uniform rectifiability

after P. Mattila, M.S. Melnikov, and J. Verdera [3]
A summary written by Joris Roos

Abstract

We discuss a necessary and sufficient condition for L^2 boundedness of the Cauchy integral on Ahlfors-David regular sets. This will imply a characterization of Ahlfors-David regular sets with vanishing analytic capacity.

11.1 Introduction

The analytic capacity of a compact set $E \subset \mathbb{C}$ is defined as $\gamma(E) = \sup\{|f'(\infty)| : f \in M_E\}$ where M_E denotes the set of bounded holomorphic functions on $\mathbb{C}\backslash E$ such that $\|f\|_\infty \le 1$ and $f(\infty) = 0$. Hence $f'(\infty) = \lim_{z\to\infty} zf(z)$ for $f \in M_E$. As was proven by Ahlfors, we have $\gamma(E) = 0$ if and only if E is *removable* in the sense that every bounded analytic function on $\mathbb{C}\backslash E$ is constant. It was a well known conjecture that for E of Hausdorff dimension 1 and $\mathcal{H}^1(E) > 0$, $\gamma(E) = 0$ holds if and only if E is *purely unrectifiable*, i.e. $\mathcal{H}^1(\Gamma \cap E) = 0$ for every rectifiable curve Γ. Here, $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. The presented article proves this in the special case that $\mathcal{H}^1(E) < \infty$ and E satisfies the regularity condition

$$M^{-1}r \le \mathcal{H}^1(E \cap B(z,r)) \le Mr \tag{1}$$

for some constant $M > 0$, $z \in E$ and $0 < r < \operatorname{diam}(E)$, where $B(z,r)$ is the closed ball of radius r around z. If condition (1) is satisfied, we will refer to E as being *Ahlfors-David regular (AD-regular)*.

Let $E \subset \mathbb{C}$ be s.t. $0 < \mathcal{H}^1(E) < \infty$. The Cauchy integral operator, symbolically given by

$$C_E f(z) = \int_E \frac{f(\zeta)}{\zeta - z}\, d\mu(\zeta),$$

is defined to be the canonical singular integral operator associated to the kernel $K(z,w) = (z-w)^{-1}$ as discussed in [1]. Here we have set $\mu = \mathcal{H}^1$.

Note that condition (1) implies that (E, μ) is a space of homogenous type in the sense of Christ. C_E is bounded in $L^2(E) = L^2(E, \mu)$ iff the family of truncated operators $(C_{E,\epsilon})_{\epsilon>0}$,

$$C_{E,\epsilon} f(z) = \int_{E \setminus B(z,\epsilon)} \frac{f(\zeta)}{\zeta - z} \, d\mu(\zeta),$$

is uniformly bounded in $L^2(E)$, i.e. with a constant independent of ϵ. The main result from the presented article [3] is

Theorem 1 (MMV)**.** *Let $E \subset \mathbb{C}$ be AD-regular. Then*

(a) C_E is bounded in L^2 if and only if E is uniformly rectifiable, and

(b) $\gamma(E) = 0$ if and only if E is purely unrectifiable.

By saying that E is *uniformly rectifiable* we mean that it is contained in an AD-regular curve. The converse of (a) follows from the T(b) theorem of Christ [1]. Regarding part (b) it suffices to show that $\gamma(E) > 0$ implies that E is not purely unrectifiable, since the other direction is already known (follows from the solution of Denjoy's conjecture). If on the other hand $\gamma(E) > 0$, then Theorem 29 in [1] and (a) imply that E is not purely unrectifiable. Hence it only remains to prove that E is uniformly rectifiable provided that C_E is bounded in L^2. We will sketch the proof of this below. The main idea is to introduce a curvature-type quantity corresponding to the measure μ using an elementary geometric insight. In [4] this was also used to prove the L^2 boundedness of the Cauchy integral on Lipschitz curves, a special case of Theorem 1 (a).

11.2 The curvature of a measure

In case $z_1, z_2, z_3 \in \mathbb{C}$ do not all lie on the same line, we define $c(z_1, z_2, z_3) = 1/R$ where R is the radius of the circle passing through z_1, z_2, z_3. Otherwise we set $c(z_1, z_2, z_3) = 0$. This quantity is called the *Menger curvature* of the triple (z_1, z_2, z_3).

Lemma 2. *Let $z \in \mathbb{C}$, $r > 0$ and $z_1, z_2, z_3 \in B(z, r)$ pairwise different. Then*

$$c(z_1, z_2, z_3) = \frac{4S(z_1, z_2, z_3)}{|z_1 - z_2||z_1 - z_3||z_2 - z_3|}, \tag{2}$$

$$c(z_1,z_2,z_3)^2 = \sum_{\sigma\in S_3} \frac{1}{(z_{\sigma(2)}-z_{\sigma(1)})\overline{(z_{\sigma(3)}-z_{\sigma(1)})}}, \text{ and} \tag{3}$$

$$\operatorname{dist}(z_1, L_{z_2,z_3}) \le 2r^2 c(z_1,z_2,z_3) \tag{4}$$

where $S(z_1,z_2,z_3)$ denotes the area of the triangle with edges z_1,z_2,z_3, $L_{z,w}$ is the line passing through $z \neq w \in \mathbb{C}$, and $\operatorname{dist}(z,L) = \inf_{w\in L}|z-w|$ for $L \subset \mathbb{C}$.

The first identity follows from elementary triangle geometry and the second can be validated by a simple computation, see [4]. Identity (2) implies (4). We turn now to the crucial lemma.

Lemma 3. *Let $E \subset \mathbb{C}$ be AD-regular and suppose that C_E is bounded in L^2. Then*

$$\iiint_{(E\cap B)^3} c(z_1,z_2,z_3)^2\, d\mu(z_1) d\mu(z_2)\, d\mu(z_3) \le Mr \tag{5}$$

for some constant $M > 0$ and B a ball of radius r.

The idea is here to apply the L^2 boundedness in the case $f = \mathbf{1}_E$ and then use Fubini and (3). Let us illustrate this by a purely symbolical calculation:

$$\begin{aligned}
\int \left| \int \frac{1}{\zeta - z} \right|^2 &= \iiint \frac{1}{(z_2-z_1)\overline{(z_3-z_1)}} \\
&= \frac{1}{6} \sum_{\sigma\in S(3)} \iiint \frac{1}{(z_{\sigma(2)}-z_{\sigma(1)})\overline{(z_{\sigma(3)}-z_{\sigma(1)})}} \\
&= \frac{1}{6} \iiint c(z_1,z_2,z_3)^2
\end{aligned} \tag{6}$$

where all the integrals are with respect to $d\mu$. The quantity on the right in (6) can be interpreted as the *curvature of the measure* μ. Making this calculation rigorous using truncations and carefully manipulating the regions over which we integrate, one obtains a proof of the lemma. It is noteworthy that the converse of Lemma 3 also holds. This can be seen from the T(1) theorem.

11.3 Sketch of the proof

Let $E \subset \mathbb{C}$ AD-regular and assume that C_E is bounded in L^2. We want to prove that E is uniformly rectifiable. To achieve this, a result by David and Semmes [2] comes in handy, which characterizes uniform rectifiability in terms of a certain quantity $\beta_2(z, r)$. Define

$$\beta_2(z,r) = \left(\inf_L r^{-3} \int_{E\cap B(z,r)} \operatorname{dist}(w,L)^2 \, d\mu(w)\right)^{\frac{1}{2}}$$

for $z \in \mathbb{C}$ and $r > 0$. where the infimum is taken over all lines in the complex plane. David and Semmes proved the following result in [2].

Theorem 4. *Let $E \subset \mathbb{C}$ be AD-regular. Then E is uniformly rectifiable if and only if there is $M > 0$ such that*

$$\int_0^R \int_{E\cap B(z,R)} r^{-1}\beta_2(w,r)^2 \, d\mu(w)dr \le MR \tag{7}$$

for all $z \in E$ and $0 < R < \operatorname{diam}(E)$.

Mattila, Melnikov and Verdera now deduce the inequality (7) from (5). The first step is to apply (4) to estimate

$$\begin{aligned} r^{-1}\beta_2(z_1,r)^2 &\le r^{-4} \int_{E\cap B(z_1,r)} \operatorname{dist}(z_2, L_{z_1,z_3})^2 d\mu(z_2) \\ &\le 4\lambda^2 \int_{E\cap B(z,2r)} c(z_1,z_2,z_3)^2 d\mu(z_2) \end{aligned}$$

for $z_1 \in B(z,r)$ and $z_3 \in B(z,\lambda r)\backslash B(z,2r)$ for some constant $\lambda > 2$. The restriction on z_3 stems from the fact that we must avoid it to be equal to z_1. Integrating with respect to z_1, z_3 now gives

$$\int_{E\cap B(z,r)} \beta_2(z_1,r)^2 d\mu(z_1) \le 4\lambda^2 \iiint\limits_{\Omega} c(z_1,z_2,z_3)^2 d\mu(z_1)d\mu(z_2)d\mu(z_3)$$

where $\Omega = E^3 \cap (B(z,r) \times B(z,2r) \times B(z,\lambda r)\backslash B(z,2r))$. After integrating from 0 to R with respect to r, this would already look a lot like the inequality we need, except for the cumbersome domain of integration on the right. However, this can be dealt with by decomposing with respect to a Vitali-type covering on $E \cap B(z,R)$, c.f. [3, Theorem 3.6].

Actually the inequalities (7) and (5) are even equivalent, as was remarked before.

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12 A T(b) theorem with remarks on analytic capacity and the Cauchy integral

after M. Christ [1]
A summary written by Sebastian Stahlhut

Abstract

The aim of M. Christ's article is to discuss testing conditions for the L^2-boundedness of singular integral operators on spaces of homogeneous type. The comparision with previous theorems by different authors as well as the case of antisymmetric kernels is in the scope of M. Christ's article. In particular, this allows applications to analytic capacity and the Cauchy integral.

12.1 Definitions

Before we start to discuss the main theorem of Michael Christ's article we need several definitions. The setting for Michael Christ's theorem are spaces of homogeneous type, or more precisely quasi-metric doubling spaces. By a quasi-metric ρ on a set X we mean a function $\rho : X \times X \to [0, \infty)$ which satisfies:

$$\begin{aligned} &\rho(x,y) = 0 && \text{iff } x = y, \\ &\rho(x,y) = \rho(y,x) && \forall x, y \in X, \\ &\rho(x,y) \leq A_0 (\rho(x,z) + \rho(z,y)) && \forall x, y, z \in X, \end{aligned}$$

where $A_0 < \infty$ is independent of x,y,z. The related balls are defined by $B(x,r) := \{y \in X : \rho(x,y) < r\}$. Then a space of homogeneous type is a set X equipped with a quasi-metric ρ such that the associated balls $B(x,r)$ are open and equipped with a nonnegative Borel measure μ satisfying the doubling condition

$$\mu(B(x,2r)) \leq A_1 \mu(B(x,r)) \qquad \forall x \in X, r > 0.$$

Here, let us also remark that the function $\lambda(x,y) := \mu(B(x, \rho(x,y)))$ will be important later on. The testing conditions under consideration are in terms of para-accretive functions or pseudo-accretive systems.

Definition 1 (Para-accretive functions). *$b \in L^{\infty}(X)$ is said to be para-accretive if there exists $\delta > 0$ such that for all $x \in X$ and all $r > 0$, there exists $x' \in B(x,r)$ and $r' \in [\delta r, r]$ such that*

$$\left| \int_{B(x',r')} b(y)\, d\mu(y) \right| \geq \delta \mu\left(B(x',r')\right).$$

By [3] the Lebesgue differentiation theorem is valid on spaces of homogeneous type. Thus there exists $\epsilon > 0$ such that $|b| \geq \epsilon$, which is important to know for the definition of singular integral operators later on. There is a weaker notion for testing conditions for singular integrals on spaces of homogeneous type.

Definition 2 (Pseudo-accretive systems). *A pseudo-accretive system is a collection of L^{∞}-functions b_B, one for each ball $B = B(x,r) \subset X$, satisfying for some $C < \infty$, $\delta > 0$*

$$||b_B||_{L^{\infty}} \leq C \quad \forall B, \qquad \textit{and} \qquad \left| \int_B b_B(y)\, d\mu(y) \right| \geq \delta \mu(B) \quad \forall B.$$

In fact, one observes that the functions functions b_B may vary here. In the following we denote by $\mathcal{D}_{\alpha}$ the space of Hölder continuous functions of order $\alpha \in (0,1]$ with compact support. Whenever $b \in L^{\infty}(X)$ satisfies $|b| \geq \epsilon$ μ-a.e. there exists an isomorphism from $b\mathcal{D}_{\alpha}$ to $\mathcal{D}_{\alpha}$. Moreover, by $(b\mathcal{D}_{\alpha})'$ one denotes the dual space of $b\mathcal{D}_{\alpha}$.

Definition 3 (Singular integral operator). *A singular integral operator T on a space of homogeneous type X is a continuous linear operator from $b_1\mathcal{D}_{\alpha}$ to $(b_2\mathcal{D}_{\alpha})'$ for some $\alpha \in (0,1]$ and some $b_1, b_2 \in L^{\infty}(X)$ satisfying $|b_1|, |b_2| \geq \delta > 0$ $\mu - a.e.$, which is associated to a standard kernel, i.e. there exists a function $K : X \times X \setminus \{x = y\} \to \mathbb{C}$ and $\epsilon, \delta > 0$, $C < \infty$ such that*

$$|K(x,y)| \leq \frac{C}{\lambda(x,y)}$$

for all distinct $x, y \in X$ and such that

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C \left(\frac{\rho(x,x')}{\rho(x,y)} \right)^{\epsilon} \frac{1}{\lambda(x,y)}$$

whenever $\rho(x,x') \leq \delta\rho(x,y)$ and the relation between operator and kernel is given by

$$\langle Tf, g\rangle_{(b_2 D_\alpha)', b_2 D_\alpha} = \int\int K(x,y) f(y) g(x) \, d\mu(x) \, d\mu(x)$$

for all $f \in b_1\mathcal{D}_\alpha$ and $g \in b_2\mathcal{D}_\alpha$.

One observes that the quantities in the definition of singular integral operators are invariant by the isomorphism from $b_i\mathcal{D}_\alpha$ to $\mathcal{D}_\alpha$, i.e. to every singular integral operator $T : b_1\mathcal{D}_\alpha \to (b_2\mathcal{D}_\alpha)'$ associated to a kernel K there exists by the isomorphism a unique corresponding singular integral operator $\widetilde{T} : \mathcal{D}_\alpha \to \mathcal{D}'_\alpha$ associated to a kernel $\widetilde{K}$. Often it is useful to conclude L^2-boundedness of singular integral operators via truncations.

Definition 4 (Truncated singular integral operator)**.** *Let T be a singular integral operator associated to a kernel K. Then we define the truncated singular integral operator $T^\epsilon : \mathcal{D}_\alpha \to (\mathcal{D}_\alpha)'$ for any α by*

$$T^\epsilon f(x) = \int_{\rho(x,y) > \epsilon} K(x,y) f(y) \, d\mu(y)$$

Besides testing conditions via para-accretive functions we will also need a weak boundedness condition. It will be a remarkable point that the weak boundedness condition is not needed in Christ's $T(b)$-theorem below.

Definition 5. *A continuous linear transformation $T : b_1\mathcal{D}_\alpha \to (b_2\mathcal{D}_\alpha)'$ is weakly bounded (with respect to b_1, b_2) if there exists $C < \infty$ such that for all $x_0 \in X$, $r > 0$ and all $\varphi_1, \varphi_2 \in B_{\alpha,x_0,r}$ holds*

$$|\langle T(b_1\varphi_1), b_2\varphi_2\rangle| \leq C\mu(B(x_0,r)).$$

Here, we denote by $B_{\alpha,x_0,r}$ the set of all $f \in \mathcal{D}_\alpha$ such that supp$f \subset \{y : \rho(x_0,y) \leq r\}$, $||f||_{L^\infty} \leq 1$ and $|f(x) - f(y)| \leq r^{-\alpha}\rho(x,y)^\alpha$ for all $x, y \in X$.

12.2 The main theorem and comparision to previous results

Now, we are in position to state Christ's theorem and to compare it to a former version by David, Journé and Semmes.

Theorem 6 (Christ's Tb theorem)**.** *Let X be a space of homogeneous type and T be a truncated singular integral operator. Suppose there exists $C < \infty$ and pseudo-accretive systems $\{b_B^1\}$ and $\{b_B^2\}$ on X such that for all B,*

$$||T\left(b_B^1\right)||_{L^\infty} \leq C, \qquad\qquad ||T^t\left(b_B^2\right)||_{L^\infty} \leq C.$$

Then T is bounded on $L^2(X,\mu)$, with an operator norm not exceeding a bound which depends only on A_0, A_1, on the bounds in the standard estimates for K, on the constants in the definition for $\{b_B^i\}$, and on C.

In comparision one has

Theorem 7 (DJS-Theorem)**.** *Suppose that b_1, b_2 are para-accretive functions and that T is a singular integral operator on a space X of homogeneous type. Suppose that T is weakly bounded from $b_1\mathcal{D}_\alpha$ to $(b_2\mathcal{D}_\alpha)'$, that α is sufficiently small, and that $T(b_1), T^t(b_2) \in BMO$. Then T extends to a bounded operator on $L^2(X,\mu)$.*

As already remarked before Christ's theorem does not need to require a weak boundedness condition. Christ's theorem permits pseudo-accretive systems instead of a single para-accretive function and it can be shown that the existence of a good pseudo-accretive system is necessary for L^2-boundedness. But, one might think that the formulation of Christ's theorem via truncated singular integrals is a disadvantage. In fact, that isn't the case. If there is a pseudo-accretive system $\{b_B\}$ such that $T(b_B) \in L^\infty$ uniformly then this pseudo-accretive system works simultanously for all truncations T^ϵ, i.e. $T^\epsilon(b_B) \in L^\infty$ uniformly in B for all $\epsilon > 0$. The consequence is that we can deduce L^2-boundedness of T by uniform L^2-boundedness of the truncations T^ϵ taking the limit $\epsilon \to 0$. The only disadvantage in relation to DJS-Theorem is that Christ's theorem requires $T(b_B^1), T^t(b_B^2) \in L^\infty$ instead of $T(b_1), T^t(b_2) \in BMO$.
An interesting case in applications is whenever the kernel is antisymmetric, i.e. $K(x,y) = -K(y,x)$. In this case the singular integral operator T can be defined for $\varphi_1, \varphi_2 \in \mathcal{D}_\alpha$, $b_1 = b_2 = b \in L^\infty$ via

$$\langle T(b\varphi_1), b\varphi_2\rangle = \frac{1}{2}\int\int K(x,y)\, b(x)\, b(y)\left[\varphi_1(y)\varphi_2(x) - \varphi_1(x)\varphi_2(y)\right] d\mu(x)\, d\mu(y)$$

and the integral converges absolutely as consequence of standard estimates and Hölder continuity of $\varphi_1, \varphi_2 \in \mathcal{D}_\alpha$. In this case of antisymmetric kernels one can always deduce L^2-boundedness of T by uniform L^2-boundedness of the truncations. So, in the case of antisymmetric kernels we get

Theorem 8 (Christ's Theorem for antisymmetric kernels)**.** *Let X be a space of homogeneous type and T be a singular integral operator associated to an antisymmetric kernel. Suppose there exists a pseudo-accrteive system $\{b_B\}$ on X such that $||T(b_B)||_{L^\infty} \leq C < \infty$ for all balls B. Then T is bounded on $L^2(X)$.*

and

Theorem 9 (DJS for antisymmetric kernels)**.** *Suppose b is an para-accretive function and T is a singular integral operator on a space X of homogeneous type . Suppose $T(b) \in BMO$. Then T is bounded on $L^2(X)$.*

The reader observes that in Christ's theorem for antisymmetric kernels we have a singular integral operator instead of a truncated singular integral operator and in DJS-Thoerem for antisymmetric kernels we are allowed to drop the weak-boundedness condition as it is automatically satisfied. This case is interesting in particular for the Cachy integral and analytic capacity. For more details we refer the reader to [1] and turn to the proof of Christ's theorem.

12.3 Dyadic cubes and the proof of Christ's theorem

The core of the proof is the following analogue of Dyadic cubes on spaces of homogeneous type.

Theorem 10 (Christ's dyadic cubes)**.** *There exists a collection of open subsets $\left\{Q_\alpha^k \subset X : k \in \mathbb{C}, \alpha \in I_k\right\}$, and constants $\delta \in (0,1)$, $a_0 > 0$, $\eta > 0$ and $C_1, C_2 < \infty$ such that*

1. $\mu\left(X \backslash \bigcup_\alpha Q_\alpha^k\right) = 0$ *for all k.*
2. *If $l \geq k$ then either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k \neq \emptyset$.*
3. *For each (k, α) and each $l < k$ there is a unique β such that $Q_\alpha^k \subset Q_\beta^l$.*
4. *Diameter $\left(Q_\alpha^k\right) \leq C_1 \delta^k$.*
5. *Each Q_α^k contains some ball $B\left(z_\alpha^k, a_0 \delta^k\right)$.*
6. $\mu\left\{x \in Q_\alpha^k : \rho\left(x, X \backslash Q_\alpha^k\right) \leq t\delta^k\right\} \leq C_2 t^\eta \mu\left(Q_\alpha^k\right)$ *for all k, α and all $t > 0$.*

Indeed this analogue of dyadic cubes allows to generalize Coifman-Jones-Semmes theorem to the setting of spaces of homogeneous type. That is

Theorem 11 (CJS-Theorem)**.** *Suppose that b^1, b^2 are dyadic para-accretive functions, that $T : b^1\mathcal{D} \to (b^2\mathcal{D})'$ is weakly bounded singular integral operator, and that $T(b^1), T^t(b^2) \in BMO$ (dyadic). Then T is bounded on L^2.*

This allows us to reduce Christ's theorem to CJS-theorem. That means having the dyadic cubes in hand one proves the following proposition.

Proposition 12 (The reduction)**.** *Let X be a SHT, let $\mathcal{Q}$ be a system of dyadic cubes on X and suppose that X itself is an element of $\mathcal{Q}$. Let T be a truncated singular integral operator. Suppose there exists a pseudo-accretive systems $\{b_B^1\}$, $\{b_B^2\}$ such that $T(b_B^1), T^t(b_B^2) \in L^\infty$ uniformly in B. Then there exists dyadic para-accretive functions b^1, b^2 such that $T(b^1), T^t(b^2) \in BMO$ (dyadic) and $T : b^1\mathcal{D}_\alpha \to (b_2\mathcal{D}_\alpha)'$ is weakly bounded for any $\alpha > 0$. Moreover, $b^1, b^2, T(b^1), T^t(b^2)$ and the constant in the weak boundedness inequality satisfy bounds depending only on A_0, A_1, on the constants in the standard estimates for K, on the constants in the definition of pseudo-accretivity for b_B^i, and on $\sup_B ||T(b_B^1)||_\infty + \sup_B ||T(b_B^1)||_\infty$.*

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13 The local $T(b)$ theorem with rough test functions

after T. Hytönen and F. Nazarov [1]
A summary written by P. Villarroya

Abstract

The authors of [1] prove a version of the local $T(b)$ theorem under minimal integrability assumptions.

13.1 Introduction

We present an outline of the current most general version of a local Tb theorem. The result solves a question posed by Hofmann in 2008 with possible applications to free boundary theory (see [3]) and stated below.

Definition 1. *(Accretive and buffered accretive systems) Let $p, q \in [1, \infty]$.*

A (p, q) accretive system for an operator T is a family of functions $(b_Q)_{Q\in\mathcal{D}}$ indexed by dyadic cubes such that

$$\operatorname{supp} b_Q \subseteq Q \ , \quad \fint_Q b_Q = 1 \ , \quad \Big(\fint_Q |b_Q|^p\Big)^{1/p} \lesssim 1 \tag{1}$$

and

$$\Big(\fint_Q |Tb_Q|^q\Big)^{1/q} \lesssim 1 \tag{2}$$

A buffered (p, q) accretive system for an operator T is a family of functions $(b_Q)_{Q\in\mathcal{D}}$ satisfying the conditions in (1), and (2) changed by

$$\Big(\fint_{2Q} |Tb_Q|^q\Big)^{1/q} \lesssim 1 \tag{3}$$

The expression that $b^1_Q \& b^2_Q$ is a $(p, q)\&(r, s)$ accretive system (or buffered accretive system) for $T_1\&T_2$ has the obvious meaning.

Theorem 2. *(Solution to Hofmann's problem). Let T be an operator associated with a C-Z kernel such that for some $p, q \in (1, \infty)$ there exist $b^1_Q \& b^2_Q$ a buffered $(p, q')\&(q, p')$ accretive system for $T\&T^*$.*

Then, $\|T\|_{L^2\to L^2}$ is bounded with constant depending only on the constants of the C-Z kernel and the constants in (1) and (3).

Remark 3. *Whether the word 'buffered' can be removed from the statement remains open for exponents such that $1/p + 1/q > 1$.*

13.2 Definitions and statements of the results

The authors' approach is based on the technique of suppressed operators used in the solution to the quantitative Vitushkin conjecture (see [2]).

Definition 4. *The suppressed singular integral operator is defined by*

$$T_\Phi f(x) = \int K_\Phi(x,y) f(y) dy \ , \qquad K_\Phi(x,y) = \frac{K(x,y)}{1 + \left(\frac{\Phi(x)\Phi(y)}{|x-y|^2}\right)^m}$$

for a suitable non-negative Lipschitz function Φ*.*

Definition 5. *Let the maximal truncated operator be defined by*

$$T_\# f(x) = \sup_{\epsilon > 0} |T_\epsilon f(x)| \ , \qquad T_\epsilon f(x) = \int_{|x-y|>\epsilon} K(x,y) f(y) dy$$

The idea is to prove, via the operator $T_\#$, a classical local $T(b)$ theorem for T_Φ and then to extrapolate boundedness to T. More explicitely: Cotlar's inequality $T_\# f \lesssim M_{q'}(Tf) + M_{v'} f$ and the bound $|T_\Phi f| \lesssim T_\# f + Mf$ allow to transfer the hypotheses from T to T_Φ, while the fact that $T_\Phi(f) = T(f)$ when $\operatorname{supp} f \subset \{\Phi = 0\}$ allows to transfer the thesis from T_Φ to T.

Definition 6. *(Ample collection; Sparse collection) We say that* $\mathscr{D}$ *is an ample collection of dyadic subcubes of a given cube* Q *with exceptional fraction* $\sigma \in (0,1)$ *if the maximal subcubes* $\tilde{Q} \subset Q$ *with* $\tilde{Q} \notin \mathscr{D}$ *satisfy* $\sum |\tilde{Q}| \le \sigma |Q|$*.*

We say that $\mathscr{D}$ *is a sparse collection of dyadic subcubes of a given cube* Q_0 *if it contains* Q_0 *and for some* $\tau > 0$ *and all* $Q \in \mathscr{D}$ *we have that the subcubes* $\tilde{Q} \in \mathscr{D}$ *with* $\tilde{Q} \subsetneq Q$ *satisfy* $\left| \bigcup \tilde{Q} \right| \le (1-\tau)|Q|$*.*

Remark 7. $\mathscr{D}$ *is a sparse collection with parameter* τ *if for all* $Q \in \mathscr{D}$ *the family* $\mathscr{D}_Q = \{Q\} \cup \{Q' \subsetneq Q : Q' \notin \mathscr{D}\}$ *is an ample collection of* Q *with exceptional fraction* $1-\tau$*.*

The cubes in a sparse collection will be often referred as stopping cubes.

Definition 8. *(Off-diagonal estimates) An accretive system* $(b_Q)_{Q \in \mathcal{D}}$ *for* T *satisfies off-diagonal estimates if for all* $\sigma > 0$*, there exist* $C_\sigma > 0$ *so that*

$$\fint_{Q'} |T(1_{(3Q')^c} b_Q)| \le C_\sigma \tag{4}$$

for all cubes Q *and all* Q' *in an ample collection of dyadic subcubes of* Q *with exceptional fraction* σ*.*

Hoffman's conjecture is an immediate consequence of this stronger result:

Theorem 9. *(Main theorem). Let T be an operator with C-Z kernel such that for some $p \in (1,\infty)$ there exist $b_Q^1 \& b_Q^2$ a $(p,p)\&(p,p)$ accretive system for $T_\# \& (T^*)_\#$ with off-diagonal estimates.*

Then, $\|T\|_{L^2 \to L^2}$ is bounded with constants depending only on the C-Z constants of the kernel and the constants in (1), (2) and (4).

If T is antisymmetric, then the hypothesis 'with off-diagonal estimates' can be dropped.

Proposition 10. *Suppose there exists b_Q a (p,p) accretive system for $T_\#$. Then, for a fixed $\rho \in (0,1)$ and a fixed cube Q there exists a non-negative function Φ with Lipschitz constant 1 such that $|\{\Phi > 0\}| \leq \rho|Q|$ and there exists b_Q^Φ an (∞, p) accretive system for T_Φ on sparse sub-cubes of Q.*

Definition 11. *(Accretive system on a sparse family) Let $p, q \in [1,\infty]$ and Q_0 be a given cube. A (p,q) accretive system for an operator T on a sparse collection $\mathscr{D}$ of subcubes of Q_0 is a family of functions $(b_Q)_{Q\in\mathscr{D}}$ such that*

$$\operatorname{supp} b_Q \subseteq Q \ , \quad \fint_{Q'} b_Q \gtrsim 1 \ , \quad \Big(\fint_{Q'} |b_Q|^p\Big)^{1/p} \lesssim 1 \tag{5}$$

and

$$\Big(\fint_{Q'} |Tb_Q|^q\Big)^{1/q} \lesssim 1 \tag{6}$$

for all $Q \in \mathscr{D}$ and all $Q' \subseteq Q$ such that Q' is not contained in any $\tilde{Q} \subsetneq Q$ with $\tilde{Q} \in \mathscr{D}$.

If $\mathscr{D}$ is sparse with parameter $\tau \in (0,1)$, then the described cubes Q' form an ample collection of subcubes of Q with exceptional fraction $1-\tau$.

Proposition 12. *(Baby Tb theorem) Let T be an operator with C-Z kernel, Q_0 be a cube and $b_Q^1 \& b_Q^2$ be a $(\infty,t)\&(\infty,t)$ accretive system for $T\&T^*$ on $\mathscr{D}_1 \& \mathscr{D}_2$, two sparse collections of subcubes of Q_0.*

We also assume that T satisfies the following weak boundedness condition:

$$|\langle T(1_Q b^1_{Q^{a,1}}), 1_Q b^2_{Q^{a,2}}\rangle| \lesssim |Q| \tag{7}$$

for all Q dyadic subcubes of Q_0 where $Q^{a,i}$ is the minimal member of $\mathscr{D}_i$ containing Q.

Then, for $s' \in (\max\{t', 2\}, \infty]$ and all $f, g \in L^{s'}(Q_0)$, we have

$$|\langle T(f), g\rangle| \lesssim \|f\|_{s'}\|g\|_{s'}|Q_0|^{1-2/s'}$$

13.3 Proofs

13.3.1 Proof of Proposition 10: construction of an (∞, u) accretive system and the Lipschitz function Φ.

Two stopping conditions. Let Q_0 be an arbitrary fixed cube. Let also $\delta, \epsilon, \sigma, \eta \in (0,1)$ be fixed parameters to be chosen later. Then,

a) We call b-**stopping cubes** to the maximal dyadic subcubes $Q \subseteq Q_0$ with $⨍_Q |b^1_{Q_0}|^p \geq C\delta^{-1}$ and denote $\mathscr{B}_1 = \mathscr{B}_1(Q_0)$ the collection of these cubes.

Let $\tilde{b}^1_{Q_0}$ be the good part of the usual Calderón-Zygmund decomposition. We also define the function $e^1_{Q_0}(x) = \sum_{Q \in \mathscr{B}_1(Q_0)} (1 + \ell(Q)^{-1}|x - c_Q|)^{-(d+\alpha)}$.

b) We call Tb-**stopping cubes** to the maximal dyadic subcubes $Q \subseteq Q_0$ satisfying either $⨍_Q |T_\# b^1_{Q_0} + Mb^1_{Q_0} + e^1_{Q_0}|^p \geq \epsilon^{-1}$, or $⨍_Q T_\#(1_{(3Q)^c} b^1_{Q_0}) > C_\sigma$, or $\left| ⨍_Q \tilde{b}^1_{Q_0} \right| \leq \eta$. We denote by $\mathscr{T}_1 = \mathscr{T}_1(Q_0)$ the collection of these cubes.

Iterating the stopping conditions. We assume that $\mathscr{B}_k$ and $\mathscr{T}_k$ are constructed. For every $Q \in \mathscr{T}_k$, we use the function b^1_Q to choose the b-stopping cubes in Q, $\mathscr{B}_1(Q)$, which we use to construct the functions $\tilde{b}^1_Q$ and e^1_Q. Then, using b^1_Q, $\tilde{b}^1_Q$ and e^1_Q, we choose the Tb-stopping cubes in Q, $\mathscr{T}_1(Q)$.

We iteratively define $\mathscr{B}_{k+1} = \bigcup_{Q \in \mathscr{T}_k} \mathscr{B}_1(Q)$ and $\mathscr{T}_{k+1} = \bigcup_{Q \in \mathscr{T}_k} \mathscr{T}_1(Q)$ These sets satisfy $\sum_{k=1}^\infty \sum_{Q \in \mathscr{B}_k(Q_0)} |Q| \leq \sum_{k=1}^\infty \delta(1-\tau)^{k-1}|Q_0| \leq \frac{\delta}{\tau}|Q_0|$ with $\tau \approx (1-\eta)^{p'}$ and the latter value is smaller than 1 for ϵ, σ, η small enough.

Finally, we define $\Phi(x) = \sup\{\mathrm{dist}(x, (3Q)^c) : Q \in \bigcup_{k=1}^\infty \mathscr{B}_k\}$ which satisfies $|\{\Phi > 0\}| \leq \rho|Q_0|$ with $\rho = \delta/\tau$ arbitrarily small for δ small enough. Moreover, $\tilde{b}^1_Q$ is a (∞, p) accretive system for T_Φ on sparse subcubes of Q.

13.3.2 Proof of the baby Tb theorem (Proposition 12).

Let Q^0 be a fixed cube and b_Q an accretive system on $\mathscr{D}$ a sparse family of Q^0. For every $Q \subset Q^0$, let Q^a be the minimal element in $\mathscr{D}$ containing Q and let Q_i be the collection of dyadic children of Q. We define

$$\mathbb{E}^b_Q(f) = \frac{\langle f \rangle_Q}{\langle b_{Q^a} \rangle_Q} 1_Q b_{Q^a} \ , \qquad \mathbb{D}^b_Q(f) = \sum_{i=1}^{2^d} \mathbb{E}^b_{Q_i}(f) - \mathbb{E}^b_Q(f)$$

to be the expectation and difference operators. The latter satisfies, for some bounded functions $\phi_{Q,i}$, the equality

$$\mathbb{D}^b_Q(f) = (\mathbb{D}^b_Q)^2(f) + \omega_Q \langle f \rangle_Q = \sum_{i=0}^{2^d} \phi_{Q,i} \langle \mathbb{D}^b_{Q,i} f \rangle_{Q_i} \tag{8}$$

with $\mathbb{D}^b_{Q,i}f = \mathbb{D}^b_Q f$ for $1 \le i \le 2^d$, $\mathbb{D}^b_{Q,i}f = \langle f\rangle_Q$ when $i = 0$ and Q has a stopping child and $\mathbb{D}^b_{Q,i}f = 0$ when $i = 0$ and Q does not have stopping child.

By the properties of the selected functions b_{Q^a}, we have for $s \in (1,\infty)$

$$\Big(\sum_{Q\subseteq Q^0}\|\mathbb{D}^b_{Q,i}f\|_2^2\Big)^{1/2} \lesssim \|f\|_2 \qquad \text{and} \qquad \Big\|\Big(\sum_{Q\subseteq Q^0}|\mathbb{D}^b_{Q,i}f|^2\Big)^{1/2}\Big\|_{L^s} \lesssim \|f\|_s \tag{9}$$

Moreover, since $f = \sum_{Q\subset Q^0}\mathbb{D}^b_Q(f)$, we write

$$\langle T(f), g\rangle = \Big(\sum_{\substack{Q,\,R\,\subseteq\,Q^0\\ \ell(Q)\,\le\,\ell(R)}} + \sum_{\substack{Q,\,R\,\subseteq\,Q^0\\ \ell(R)\,<\,\ell(Q)}}\Big)\langle T\mathbb{D}^{b^1}_Q f, \mathbb{D}^{b^2}_R g\rangle$$

By symmetry, we just need to study the first term, which we parametrize as

$$\sum_{k=0}\sum_{m\in\mathbb{Z}^d}\sum_{R\subseteq Q^0}\langle T\mathbb{D}^{b^1,k}_S f, \mathbb{D}^{b^2}_R g\rangle \tag{10}$$

with $S = R + m\ell(R)$, $\mathbb{D}^{b^1,k}_S f = \sum\limits_{\substack{Q\,\subset\,S\\ \ell(Q)\,=\,2^{-k}\ell(S)}}\mathbb{D}^{b^1}_Q f$. We analyse (10) in separate cases.

1. Disjoint cubes. This case assumes $|m| \ge 1$ and so, we can write

$$\langle T\mathbb{D}^{b^1,k}_S f, \mathbb{D}^{b^2}_R g\rangle = \sum_{i,j}\int_{R\times S}\mathbb{D}^{b^1,k}_{S,i}f(y)K^{i,j;k}_{R,S}(x,y)\mathbb{D}^{b^2,k}_{R,j}g(x)dydx$$

with

$$K^{i,j;k}_{R,S}(x,y) = \sum_{\substack{Q\,\subset\,S\\ \ell(Q)\,=\,2^{-k}\ell(S)}}\frac{1_{Q_i}(y)}{|Q_i|}\langle T\phi^1_{Q,i}, \phi^2_{R,j}\rangle\frac{1_{R_j}(x)}{|R_j|}$$

It is proved that $\|K^{i,j;k}_{R,S}\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \lesssim 2^{-k\min\{\alpha,1/2\}}(1+k)^{\delta_{\alpha,1/2}}|m|^{-(d+\alpha)}$ with $\delta_{\alpha,1/2}$ the Kronecker's delta. This estimate suffices to bound the corresponding terms in (10) by $C\|f\|_2\|g\|_2$.

2. Nested cubes. In this case, $m = 0$ and we perform the decomposition

$$\sum_{k=0}^{\infty}\sum_R\langle T\mathbb{D}^{b^1,k}_R f, \mathbb{D}^{b^2}_R g\rangle = \sum_R\langle T\mathbb{D}^{b^1}_R f, \mathbb{D}^{b^2}_R g\rangle + \sum_R\sum_{Q\subsetneq R}\langle T\mathbb{D}^{b^1}_Q f, \mathbb{D}^{b^2}_R g\rangle = D + ND$$

2.1. The **non-diagonal part** is proved to satisfy the decomposition

$$ND = \sum_Q\frac{\langle T\mathbb{D}^{b^1}_Q f, b^2_{Q^{a,2}}\rangle\langle g\rangle_Q}{\langle b^2_{Q^{a,2}}\rangle_Q} + \sum_{k=0}^{\infty}\sum_R\sum_{\substack{S\,\subset\,R\\ \ell(S)\,=\,\ell(R)/2}}\sum_{j=0}^{2^d}\langle T\mathbb{D}^{b^1,k}_S f, 1_{S^c}\psi^{b^2}_{R,j;S}\rangle\langle\mathbb{D}^{b^2}_{R,j}g\rangle_{R_j} = P + NP$$

for some bounded functions $\psi^{b^2}_{R,j;S}$. The **second term** can be treated with the same ideas as the disjoint cubes (case 1).

On the other hand, the **paraproduct** is dealt with (8) and an L^s version of the Carleson embedding theorem:

$$P \lesssim \sum_{R\subseteq Q^0} \langle \mathbb{D}^{b^1}_R f, (\mathbb{D}^{b^1}_R)^* T^* b^2_{R^{a,2}}\rangle \langle g\rangle_R| + \sum_{R\subseteq Q^0} |\langle \mathbb{D}^{b^1}_{R,0} f, \omega^1_R T^* b^2_{R^{a,2}}\rangle \langle g\rangle_R|$$

$$\lesssim \Big\|\Big(\sum_{R\subseteq Q^0} |\mathbb{D}^{b^1}_R f|^2\Big)^{1/2}\Big\|_{L^{s'}} \Big\|\Big(\sum_{R\subseteq Q^0} |(\mathbb{D}^{b^1}_R)^* T^*(b^2_{R^{a,2}})\langle g\rangle_R|^2\Big)^{1/2}\Big\|_{L^s} + ST$$

$$\lesssim \|f\|_{L^{s'}}\|g\|_{L^s} \sup_{S\subseteq Q^0} |S|^{-1/s} \Big\|\Big(\sum_{R\subseteq S} |(\mathbb{D}^{b^1}_R)^* T^*(b^2_{R^{a,2}})|^2\Big)^{1/2}\Big\|_{L^s} + ST'$$

where the terms ST and ST' are similar to their preceding ones (changing $(\mathbb{D}^{b^1}_R)^*$ by ω_R, and $\mathbb{D}^{b^1}_R f$ by $\mathbb{D}^{b^1}_{R,0} f$) and can be bounded using similar ideas.

Now, we decompose all subcubes $P\subseteq S$ into k-th generations of maximal subcubes so that $P = P^{a,2}$, which we denote by $\mathcal{P}^k(S)$. Then, by disjointness of the elements in $\mathcal{P}^k(S)$ and (9), the $\|\|_{L^s}$-norm is bounded by

$$\sum_{k=0}^{\infty}\Big(\sum_{P\in\mathscr{P}_k(S)} \Big\|\Big(\sum_{\substack{R\,\subseteq\, P\\ R^{a,2}\,=\,P^{a,2}}} |(\mathbb{D}^{b^1}_R)^* T^*(b^2_{R^{a,2}})|^2\Big)^{1/2}\Big\|^s_{L^s}\Big)^{1/s} \lesssim \sum_{k=0}^{\infty}\Big(\sum_{P\in\mathcal{P}^k(S)} \|1_P T^* b^2_{P^{a,2}}\|^s_{L^s}\Big)^{1/s}$$

$$\lesssim \sum_{k=0}^{\infty}\Big(\sum_{P\in\mathcal{P}^k(S)} |P|\Big)^{1/s} \lesssim \sum_{k=0}^{\infty}\big((1-\tau)^{k-1}|S|\big)^{1/s} \lesssim |S|^{1/s}$$

ending this case.

2.2. For the **diagonal term**, since $\mathbb{D}^{b^1}_R = \sum_{i=1}^{2^d} 1_{R_i}\mathbb{D}^{b^1}_R$, we have that

$$D \lesssim \sum_R \sum_{i,j:i\neq j} \|1_{R_i}\mathbb{D}^{b^1}_R f\|_2 \|1_{R_j}\mathbb{D}^{b^2}_R g\|_2 + \sum_R \sum_{j=1}^{2^d} |\langle T(1_{R_j}\mathbb{D}^{b^1}_R f), 1_{R_j}\mathbb{D}^{b^2}_R g\rangle| \quad (11)$$

where the first term follows from Hardy's inequality and it is easily bounded by $C\|f\|_2\|g\|_2$. To deal with the last part, it is first proved that

$$|\langle T(1_{R_j}\mathbb{D}^{b^1}_R f), 1_{R_j}\mathbb{D}^{b^2}_R g\rangle| \lesssim \sum_{i,h\in\{0,j\}} |\langle T(1_{R_j} b^1_{R^{a,1}_i}), 1_{R_j} b^1_{R^{a,1}_h}\rangle| \sum_{i,h\in\{0,j\}} |\langle \mathbb{D}^{b^2}_{R,i} f\rangle_{R_i}||\langle \mathbb{D}^{b^1}_{R,h} g\rangle_{R_h}|$$

By the weak boundedness property (7), the first factor is dominated by $|R|$. Then, with inequality $|R|^{1/2}|\langle \mathbb{D}^{b^1}_{R,i} f\rangle_{R_i}| \le \|\mathbb{D}^{b^1}_{R,i} f\|_2$ and (9), the second term in (11) is bounded by $C\|f\|_2\|g\|_2$. This finishes the proof of Proposition 12.

13.3.3 Proof of the main result (Theorem 9).

The hypotheses and Cotlar's inequality imply the existence of $b^1_Q \& b^2_Q$ a $(p,p)\&(p,p)$ accretive system for $T_\# \& (T_\#)^*$ satisfying the hypotheses of Proposition 10. Then, there exist a Lipschitz function Φ and $\tilde{b}^1_Q \& \tilde{b}^2_Q$ a $(\infty,p)\&(\infty,p)$ accretive system for $T_\Phi \& (T_\Phi)^*$ on sparse cubes of Q_0 satisfying the hypotheses of Proposition 12. This implies that for $s' \in (p',\infty]$ and all $f,g \in L^{s'}(Q_0)$,

$$|\langle T_\Phi(f), g\rangle| \lesssim \|f\|_{s'}\|g\|_{s'}|Q_0|^{1-2/s'} \tag{12}$$

Let $b_{Q_0} = \frac{|Q_0|}{|Q_0\cap\{\Phi=0\}|}1_{Q_0\cap\{\Phi=0\}}$ which satisfies: $⨍_{Q_0} b_{Q_0} = 1$, $\|b_{Q_0}\|_\infty \lesssim 1$. Moreover, $\operatorname{supp} b_{Q_0} \subset \{\Phi = 0\}$ and so, $T_\Phi b_{Q_0} = Tb_{Q_0}$ and $T^*_\Phi b_{Q_0} = T^* b_{Q_0}$.

Applying (12) to $f = b_{Q_0}$ and an arbitrary g with $\|g\|_{L^{s'}(Q_0)} = 1$, we get

$$\Big(\int_{Q_0} |Tb_{Q_0}|^s\Big)^{1/s} \lesssim |Q_0|^{1/s}$$

and similar for T^* (choosing $g = b_{Q_0}$ and an arbitrary f with $\|f\|_{L^{s'}(Q_0)} = 1$).

Then, $b_{Q_0}\&b_{Q_0}$ is a $(\infty,s)\&(\infty,s)$ accretive system for $T\&T^*$. By a standard stopping time construction, for every cube Q_0 we can extract a $(\infty,s)\&(\infty,s)$ accretive system for $T\&T^*$ on sparse subcubes of Q_0.

By Proposition 12 again, for $r' \in (\max\{r',2\},\infty]$ and all $f,g \in L^{r'}(Q_0)$,

$$|\langle T(f), g\rangle| \lesssim \|f\|_{r'}\|g\|_{r'}|Q_0|^{1-2/r'} \tag{13}$$

Now, using (13) first for $f = 1_{Q_0}$ and an arbitrary g with $\|g\|_{L^{s'}(Q_0)} = 1$ and later for $g = 1_{Q_0}$ and an arbitrary f with $\|f\|_{L^{s'}(Q_0)} = 1$, we deduce

$$\Big(\int_{Q_0} |T1_{Q_0}|^r\Big)^{1/r} \lesssim |Q_0|^{1/r} \quad , \qquad \Big(\int_{Q_0} |T^*1_{Q_0}|^r\Big)^{1/r} \lesssim |Q_0|^{1/r}$$

Finally, the standard local $T(1)$ theorem proves the $\|T\|_{L^2\to L^2}$ boundedness.

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14 Two short proofs of L^2 boundedness for the Cauchy integral operator on Lipschitz curves

after R. R. Coifman, Peter W. Jones, and Stephen Semmes [1]
A summary written by Marco Vitturi

Abstract

We give two short proofs of the $L^2 \to L^2$ boundedness of the Cauchy integral operator on Lipschitz curves. The two proofs are similar in spirit but rely on different techniques, the first one using complex analysis, the second one using a suitably modified Haar basis. The second proof can be modified to yield the proof of a $T(b)$ theorem.

14.1 Introduction

Let Γ be a Lipschitz graph embedded in $\mathbb{C}$, that is $\Gamma = \{x + iA(x) : x \in \mathbb{R}\}$, where A is a Lipschitz function. The Cauchy integral is defined for functions $f : \mathbb{C} \to \mathbb{C}$ and points $z \in \Omega^+ := \{x + iA(x) + iy : x \in \mathbb{R}, y > 0\}$ (the epigraph) as

$$\mathcal{C}f(z) := \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{z - \zeta}\, d\zeta.$$

From this, one defines it for points on Γ itself by taking the limit

$$\mathcal{C}f(z) := \lim_{\delta \to 0^+} \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{z + i\delta - \zeta}\, d\zeta \qquad z \in \Gamma.$$

Formally, going back to real variables, one can write

$$\mathcal{C}f(x) = \frac{1}{2\pi i} \int_\mathbb{R} \frac{[f(y)(1 + iA'(y))]}{(x + iA(x)) - (y + iA(y))}\, dx,$$

where the integral is to be interpreted carefully.
The Cauchy integral is interesting because it's an example (probably the naivest one) of a singular integral operator - satisfying the so-called standard estimates - which is not a convolution operator. Its $L^2 \to L^2$ boundedness was first addressed by Calderón in [2], where it was proven under the weaker condition that the Lipschitz constant $\|A'\|_{L^\infty}$ be small.

The theorem addressed in the paper is

Theorem 1. *Let Γ be a Lipschitz curve in $\mathbb{C}$ as defined before and let f be a function in $L^2(\Gamma, ds)$,where ds is the arc length. Then there exists a constant $C > 0$ such that*

$$\|\mathcal{C}f\|_{L^2(\Gamma,ds)} \leq C\|f\|_{L^2(\Gamma,ds)} \tag{1}$$

for every such f. Moreover, the constant C only depends on $\|A'\|_{L^\infty}$.

As stated in the abstract, the article provides two different proofs - although similar in spirit. We outline them in the next sections.

14.2 First proof: complex analysis

It is based on two lemmas. We first introduce a norm $\|\cdot\|_{\Omega^+}$ induced by an hermitian product on the measurable $\mathbb{C}$-valued functions on Ω^+:

$$\|f\|_{\Omega^+} := \int_{\Omega^+} |f(z)|^2 d(z)\, dx\, dy,$$

where $d(z) = \operatorname{dist}(z, \Gamma)$ and x, y are real and imaginary part of z. The associated Hilbert space is denoted by $\mathcal{H}(\Omega^+)$.

Lemma 2. *Let F be a holomorphic function in Ω^+ such that $F \to 0$ at infinity. Then*

$$\|F\|_{L^2(\Gamma)} \lesssim \|A'\|_{L^\infty}\|F'\|_{\Omega^+} \tag{2}$$

Allowing the control of a certain L^2 "boundary" norm of a function through its derivative, this lemma is in the spirit of Littlewood-Paley theory. To prove it, one notices that Ω^+ is conformally equivalent to $\mathbb{H}^+$ (the upper half-plane) via a conformal map Φ, and then uses this Φ for a change of variables. Köbe's and Schwarz's complex analysis lemmas also come into play yielding useful estimates.

Lemma 3. *Let T be the operator*

$$Tg(z) := \int_{\Omega^+} \frac{g(\zeta)d(\zeta)}{(z-\zeta)^2} dx\, dy.$$

Then T is $\mathcal{H}(\Omega^+) \to L^2(\Gamma)$ bounded:

$$\|Tg\|_{L^2(\Gamma)} \lesssim \|A'\|_{L^\infty}\|g\|_{\Omega^+}. \tag{3}$$

The proof of this lemma is just tecnical, but it's worth mentioning it's obtained via Schur's test.
With these two lemmas one can easily prove theorem 1, as follows.

$$\|\mathcal{C}f\|_{L^2(\Gamma)} \lesssim \|A'\|_{L^\infty}\|(\mathcal{C}f)'\|_{\Omega^+} = \|A'\|_{L^\infty} \sup_{\|g\|_{\Omega^+}\leq 1} |\langle(\mathcal{C}f)', g\rangle_{\Omega^+}|,$$

by lemma 2, and

$$|\langle(\mathcal{C}f)', g\rangle_{\Omega^+}| = \left|\int_{\Omega^+} (\mathcal{C}f)'(z)\overline{g(z)}d(z)\,dx\,dy\right| = \left|\int_{\Omega^+}\int_\Gamma \frac{f(\zeta)\overline{g(z)}}{(z-\zeta)^2}d(z)\,d\zeta\,dx\,dy\right|$$

$$= \left|\int_\Gamma f(\zeta)T\overline{g}(\zeta)\,d\zeta\right| \leq \|f\|_{L^2(\Gamma)}\|T\overline{g}\|_{L^2(\Gamma)} \lesssim \|A'\|_{L^\infty}\|f\|_{L^2(\Gamma)}\|g\|_{\Omega^+} \lesssim_{A'} \|f\|_{L^2(\Gamma)},$$

by lemma 3, where we used Fubini and Cauchy-Schwarz in the end.

14.3 Second proof: Haar-like basis

This proof is again based on two lemmas. Using the parametrization of Γ one reduces to study operator

$$Tf(y) = \lim_{\delta\to 0^+}\int_{\mathbb{R}} \frac{f(x)}{z(y)+i\delta - z(x)} z'(x)\,dx,$$

where $z(x)$ is assumed to be an arc-length parametrization of Γ. With $I \in \mathfrak{F}$, the collection of dyadic intervals, one introduces the modified wavelets

$$\psi_I(x) := \frac{1}{|I|^{1/2}}\left(\frac{m_I^\ell m_I^r}{m_I}\right)^{1/2}\left(\frac{\chi_I^\ell(x)}{m_I^\ell} - \frac{\chi_I^r(x)}{m_I^r}\right),$$

where $m_I := |I|^{-1}\int_I z'(x)\,dx$ and I^ℓ, I^r are respectively the left and right halves of I. Then one introduces the bilinear product

$$\langle f, g\rangle_\Gamma := \int_{\mathbb{R}} f(x)g(x)\,z'(x)\,dx$$

and proves

Lemma 4. *$\{\psi_I\}_{I\in\mathfrak{F}}$ is an orthonormal basis for $(L^2(\mathbb{R}), \langle\cdot,\cdot\rangle_\Gamma)$ and*

$$\sum_{I\in\mathfrak{F}} |\langle f, \psi_I\rangle_\Gamma|^2 \sim \|f\|_{L^2(\mathbb{R})}.$$

Again this falls within Littlewood-Paley theory. The proof requires the introduction of an expectation operator $\mathbb{E}$ with respect to the complex measure $z'(x)\,dx$,

$$\mathbb{E}_j f(x) := \frac{1}{|I| m_I} \int_I f z'\, dy \qquad \text{when } x \in I, |I| = 2^{-j},$$

and the difference operator

$$\Delta_j f(x) := \mathbb{E}_{j+1} f(x) - \mathbb{E}_j f(x).$$

Then one proves that $\Delta_j = \sum_{I \in \mathfrak{F}, |I| = 2^{-j}} \langle \psi_I, \cdot \rangle_\Gamma \psi_I$ (a projection), so that $\mathbb{I} \overset{L^2}{=} \sum_{j \in \mathbb{Z}} \Delta_j$ ($\mathbb{I}$ is the identity). An application of Carleson's theorem on Carleson measures yields the size equivalence.

Lemma 5. *With T as above, one has*

$$\sup_{I \in \mathfrak{F}} \sum_{J \in \mathfrak{F}} \left(|\langle T\psi_I, \psi_J \rangle_\Gamma| + |\langle T\psi_J, \psi_I \rangle_\Gamma| \right) < \infty.$$

This is nothing but an adapted Schur's test. It's proven by scaling (so that one can take $I = [0,1]$ and remove the supremum) and by carefully estimating the contribution of every term through elementary estimates on the size of $|T(\psi_I)(x)|$ when x is far from I itself, and when it's close or within it. Alternatively, one can exploit the property $\Delta_j^2 = \Delta_j \Rightarrow \mathbb{I} = \sum_{j \in \mathbb{Z}} \Delta_j^2$, writing $T = \sum_{j,k \in \mathbb{Z}} \Delta_k \left(\Delta_k T \Delta_j \right) \Delta_j$ and reducing to estimate instead

$$\sup_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \|\Delta_k T \Delta_j\|_{L^2 \to L^2} + \sup_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \|\Delta_k T \Delta_j\|_{L^2 \to L^2}.$$

14.4 Remarks on a $T(b)$ theorem

Let b be a dyadic pseudo-accretive function, i.e. there exists $\delta > 0$ s.t.

$$\left| \frac{1}{|I|} \int_I b\, dx \right| \geq \delta \qquad \text{for every } I \in \mathfrak{F}.$$

If the standard operator T satysfies the weak boundedness property and $T(b) = T^t(b) = 0$, then one can prove it is $L^2 \to L^2$ bounded by writing

$$T = \sum_{j,k \in \mathbb{Z}} \Delta_k \left(\Delta_k T M_b \Delta_j \right) \Delta_j M_b^{-1},$$

with $M_b f := b \cdot f$, and doing an analogue of what outlined at the end of the previous section. If b is (dyadic) para-accretive only, i.e. there exists $\delta, \varepsilon > 0$ s.t. for every I there exists $I' \subset I$ s.t. $|I'| \geq \varepsilon |I|$ and

$$\left| \frac{1}{|I'|} \int_{I'} b \, dx \right| \geq \delta,$$

then one has to change accordingly the σ-algebra relative to which the conditional expectation $\mathbb{E}_j$ is taken (it was the one generated by dyadic intervals of length 2^{-j} before) in order for the above machinery to work in this case as well.

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15 Boundedness of the twisted paraproduct

after V. Kovac [1]
A summary written by Michal Warchalski

Abstract

We prove L^p estimates for a two-dimensional bilinear dyadic and continous operator of paraproduct type.

15.1 Introduction and notation

We denote dyadic martingale averages and differences by

$$\mathbb{E}_k f = \sum_{|I|=2^{-k}} \left(\frac{1}{|I|} \int_I f \right) \mathbb{1}_I, \quad \Delta_k = \mathbb{E}_{k+1} - \mathbb{E}_k, \quad k \in \mathbb{Z}$$

and the sum is taken over dyadic intervals $I \subset \mathbb{R}$ of length 2^{-k}. When we apply operator of two-dimensional function in only one variable, we mark that variable in the superscript. For instance,

$$(\mathbb{E}_k^{(1)} F)(x, y) := (\mathbb{E}_k F(\cdot, y))(x).$$

Then we can define the dyadic twisted paraproduct as

$$T_d(F, G) = \sum_{k \in \mathbb{Z}} (\mathbb{E}_k^{(1)} F)(\Delta_k^{(2)} G).$$

For two functions $\varphi, \psi \in C^1(\mathbb{R})$ satisfying

$$|\partial^j \varphi(x)| \lesssim (1 + |x|)^{-3}, \quad |\partial^j \psi(x)| \lesssim (1 + |x|)^{-3}, \quad \text{for} \quad j = 0, 1,$$

$$\mathrm{supp}(\hat{\psi}) \subset \left\{ \xi \in \mathbb{R} : \frac{1}{2} \leq |\xi| \leq 2 \right\}$$

we define the continous twisted paraproduct as

$$T_c(F, G) = \sum_{k \in \mathbb{Z}} (P_{\varphi_k} F)(P_{\psi_k} G),$$

where $P_\varphi f = f * \varphi$.

The main result of the paper is the following theorem.

Theorem 1. *(a) Operators T_d and T_c satisfy the strong bound*

$$\|T(F,G)\|_{L^{pq/(p+q)}(\mathbb{R}^2)} \lesssim_{p,q} \|F\|_{L^p(\mathbb{R}^2)}\|G\|_{L^q(\mathbb{R}^2)},$$

if $1 < p,q < \infty$, $\frac{1}{p}+\frac{1}{q} > \frac{1}{2}$.

(b) Operators T_d and T_c satisfy the weak bound

$$\alpha\left|\left\{(x,y)\in\mathbb{R}^2 : |T(F,G)(x,y)| > \alpha\right\}\right|^{(p+q)/pq} \lesssim_{p,q} \|F\|_{L^p(\mathbb{R}^2)}\|G\|_{L^q(\mathbb{R}^2)},$$

if $p = 1$, $1 \le q < \infty$ or $q = 1$, $1 \le p < \infty$.

(c) The weak estimate fails for $p = \infty$ or $q = \infty$.

Our strategy is to prove (a) for operator T_d for $p,q > 2$ and then extend the range of exponents proving the weak estimate (b) and using real multilinear interpolation. We establish bounds for T_c relating T_c to T_d. In the final step we discuss counterexample for (c). Since we dualize, we are concerned with the proper trilinear form

$$\Lambda_d(F,G,H) := \int_{\mathbb{R}^2} T_d(F,G)(x,y)H(x,y)dxdy.$$

For a dyadic interval I we denote the Haar scaling function and the Haar wavelet by

$$\varphi_I^d := |I|^{-1/2}\mathbb{1}_I, \quad \psi_I^d := |I|^{-1/2}(\mathbb{1}_{I_{left}} - \mathbb{1}_{I_{right}})$$

respectively. We can rewrite martingale averages and differences in the Haar basis as

$$\mathbb{E}_k f = \sum_{|I|=2^{-k}} \left(\int_{\mathbb{R}} f\varphi_I^d\right)\varphi_I^d, \quad \Delta_k f = \sum_{|I|=2^{-k}} \left(\int_{\mathbb{R}} f\psi_I^d\right)\psi_I^d.$$

Thus, we can rewrite the twisted paraproduct and the trilinear form as

$$T_d(F,G) = \sum_{I\times J\in\mathcal{C}} \int_{\mathbb{R}^2} F(u,y)G(x,v)\varphi_I^d(u)\varphi_I^d(x)\psi_J^d(v)\psi_J^d(y)dudv,$$

$$\Lambda_d(F,G,H) = \sum_{I\times J\in\mathcal{C}} \int_{\mathbb{R}^4} F(u,y)G(x,v)H(x,y)$$
$$\varphi_I^d(u)\varphi_I^d(x)\psi_J^d(v)\psi_J^d(y)dudxdvdy,$$

where $\mathcal{C}$ denotes the collection of all dyadic squares. We also introduce the Gowers box inner-product for functions F_1, F_2, F_3, F_4 and a dyadic square $Q = I \times J$ as

$$[F_1,F_2,F_3,F_4]_{\Box(Q)} := \frac{1}{|Q|^2}\int_I\int_I\int_J\int_J F_1(u,v)F_2(x,v)$$
$$F_3(u,y)F_4(x,y)dudxdvdy,$$

which induces the Gowers box norm

$$\|F\|_{\Box(Q)} := [F,F,F,F]_{\Box(Q)}^{1/4}.$$

15.2 Telescoping identities over trees

As mentioned above, we start by proving (a) for a certain range of exponents. We also reduce our argument to show the bound for nonnegative dyadic step functions.

We call a tree a subset $\mathcal{T}$ of dyadic squares $\mathcal{C}$ if it satisfies the following condition: there exists $Q_{\mathcal{T}} \in \mathcal{T}$, called the root of $\mathcal{T}$ that satisfies $Q \subset Q_{\mathcal{T}}$ for any $Q \in \mathcal{T}$. A tree is convex if for every $Q_1, Q_3 \in \mathcal{T}$ the inclusions $Q_1 \subset Q_2 \subset Q_3$ imply $Q_2 \in \mathcal{T}$. A leaf is a square which is not an element of $\mathcal{T}$, but its parent is. We denote by $\mathcal{L}(\mathcal{T})$ the collection of leaves of a tree $\mathcal{T}$. For any tree $\mathcal{T}$ we have corresponding version of the form Λ_d

$$\Lambda_{\mathcal{T}}(F,G,H) = \sum_{I\times J\in\mathcal{T}} \int_{\mathbb{R}^4} F(u,y)G(x,v)H(x,y)$$
$$\varphi_I^d(u)\varphi_I^d(x)\psi_J^d(v)\psi_J^d(y)dudxdvdy.$$

It is more convenient to introduce quadrilinear forms

$$\Theta^{(2)}_{\mathcal{T}}(F_1,F_2,F_3,F_4) := \sum_{I\times J\in\mathcal{T}} \int_{\mathbb{R}^4} F_1(u,v)F_2(x,v)F_3(u,y)F_4(x,y)$$
$$\varphi^d_I(u)\varphi^d_I(x)\psi^d_J(v)\psi^d_J(y)dudvdxdy,$$

$$\Theta^{(1)}_{\mathcal{T}}(F_1,F_2,F_3,F_4) := \sum_{I\times J\in\mathcal{T}} \sum_{j\in\{left,right\}} \int_{\mathbb{R}^4} F_1(u,v)F_2(x,v)F_3(u,y)F_4(x,y)$$
$$\psi^d_I(u)\psi^d_I(x)\varphi^d_{J_j}(v)\varphi^d_{J_j}(y)dudvdxdy.$$

Note that $\Lambda_{\mathcal{T}}(F,G,H) := \Theta^{(2)}_{\mathcal{T}}(\mathbb{1},G,F,H)$. We also denote for any $\mathcal{F}\subset\mathcal{C}$:

$$\Xi_{\mathcal{F}}(F_1,F_2,F_3,F_4) := \sum_{I\times J\in\mathcal{F}} \int_{\mathbb{R}^4} F_1(u,v)F_2(x,v)F_3(u,y)F_4(x,y)$$
$$\varphi^d_I(u)\varphi^d_I(x)\varphi^d_J(v)\varphi^d_J(y)dudvdxdy = \sum_{Q\in\mathcal{F}} |Q|[F_1,F_2,F_3,F_4]_{\square(Q)}.$$

After introducing essential definitions we can formulate the following lemma which is crucial for the proof.

Lemma 2. *(Telescoping identity) For any finite convex tree $\mathcal{T}$ with root $Q_{\mathcal{T}}$ we have*

$$\Theta^{(1)}_{\mathcal{T}}(F_1,F_2,F_3,F_4) + \Theta^{(2)}_{\mathcal{T}}(F_1,F_2,F_3,F_4)$$
$$= \Xi_{\mathcal{L}(\mathcal{T})}(F_1,F_2,F_3,F_4) - \Xi_{Q_{\mathcal{T}}}(F_1,F_2,F_3,F_4).$$

The telescoping identity leads to the so-called *single tree estimate*:

Proposition 3. *For any finite convex tree $\mathcal{T}$ we have*

$$|\Theta^{(2)}_{\mathcal{T}}(F_1,F_2,F_3,F_4)| \le 2|Q_{\mathcal{T}}| \prod_{j=1}^{4} \max_{Q\in\mathcal{L}(\mathcal{T})} \|F_j\|_{\square(Q)}.$$

In particular

$$|\Lambda_{\mathcal{T}}(F,G,H)| \le 2|Q_{\mathcal{T}}| \max_{Q\in\mathcal{L}(\mathcal{T})} \|F\|_{\square(Q)} \max_{Q\in\mathcal{L}(\mathcal{T})} \|G\|_{\square(Q)} \max_{Q\in\mathcal{L}(\mathcal{T})} \|H\|_{\square(Q)}.$$

Using this proposition we are able to derive the point (a) of the theorem for $2 < p, q < \infty$. Next, we use one-dimensional Cálderon-Zygmund decomposition for $F(\cdot, y), G(x, \cdot)$ for every $x, y \in \mathbb{R}$ to obtain the bound

$$\|T(F, G)\|_{L^{p/(p+1),\infty}(\mathbb{R}^2)} \lesssim_p \|F\|_{L^p(\mathbb{R}^2)} \|G\|_{L^1(\mathbb{R}^2)},$$

what is exactly the point (b) of the theorem for the operator T_d. Finally, we interpolate to get (a) in the whole range $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$, $1 < p, q < \infty$.

15.3 The continous case

The key tool of the transition to the continous case is the following result due to Jones, Seeger and Wright[2].

Proposition 4. *Let the function φ be as in the beginning and additionally let $\int_{\mathbb{R}} \varphi = 1$. The square function*

$$\mathcal{S}_{JSW,\varphi} := \left(\sum_{k \in \mathbb{Z}} |P_{\varphi_k} f - \mathbb{E}_k f|^2 \right)^{1/2}$$

is bounded from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ for $1 < p < \infty$, with the constant depending only on p.

Combining this proposition with already proven bound in the dyadic case we show (a) for the operator T_c.

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16 A boundedness criterion for generalized Caldern-Zygmund operators

after G. David and J-L. Journé [1]
A summary written by Jingxin Zhong

Abstract

We outline a proof of T1 theorem due to G. David and J-L. Journé.

16.1 Introduction

Singular integral operators appear naturally in analysis. For the classical singular integral operators of convolution type, such as Hilbert transform, the fourier transform method gives a satisfatory description. For more general singular integral operators, the classical fourier method is no longer effective and a more sophisticated method is needed. The class of Calderón-Zygmund operator (CZO) is an important generalization of the classical singular integral operators, whose distributional kernels still satisfy some regularity conditions. To define a CZO, we first need the following notion:

Definition 1. *A standard kernel is a continuous function K defined on $\Omega = R^n \times R^n \backslash \Delta$, where $\Delta = \{(x, y); x = y\}$, such that there exist two constants $\delta \in (0, 1]$ and $C > 0$, and K satisfies the following conditions:*
For all $(x, y) \in \Omega$,

$$|K(x, y)| \leq C|x - y|^{-n}, \tag{1}$$

For all x, x', y such that $|x' - x| < \frac{1}{2}|x - y|$,

$$|K(x', y) - K(x, y)| + |K(y, x') - K(y, x)| \leq C\frac{|x' - x|^\delta}{|x - y|^{n+\delta}}. \tag{2}$$

Definition 2. *A Calderón-Zygmund operator is a bounded operator from the class $S(R^n)$ of Schwartz functions to its dual $S'(R^n)$ associated with a standard kernel K such that:*
1) For all functions $f, g \in C_c^\infty(R^n)$ with disjoint supports, $< Tf, g >= \iint K(x, y)f(y)g(x)dxdy$, where $<, >$ is the dual paring of $S'(R^n)$ and $S(R^n)$;
2) T can be extended to a bounded operator on $L^2(R^n)$.

Actually from the defintion of CZOs, we can easily see that the adjoint operator T^* defined by $< T^*g, f >=< Tf, g >$ is also a CZO associated with the kernel $K^*(x, y) = K(y, x)$. If we work a little harder, we can establish the boundedness of CZOs in various L^p space for $1 < p \leq \infty$. Given the nice properties of CZOs, the problem is to give a workable criterion for CZOs. In general, one can write down the kernel of a singular integral operator explicitly and check the standard kernel condition. However the L^2-boundedness condition seems impossible to verify. The David and Journé $T1$ theorem, which is the theme of this summary, gives a necessry, sufficient and workable criterion for the L^2-boundedness, which roughly states that one can verify the L^2-boundedness condition of a singular integral operator with a standard kernel simply by checking its action on the constant function 1.

16.2 Preliminaries

To state the T1 theorem, we first need to define the action of a CZO, T on 1, or more generally on any bounded smooth function. The result will be a distribution (up to some modulation) acting on functions in a dense subspace of the Hardy space H^1.

Let $f \in C^\infty(R^n)$ be bounded, $g \in C_c^\infty(R^n)$ with integral 0, $f_1 \in C_c^\infty(R^n)$ coincide with f in a neighborhood of $supp(g)$ and $f_2 = f - f_1$. We define $< Tf, g >=< Tf_1, g > + \iint K(x, y)f_2(y)g(x)dxdy$. Note that $< Tf_1, g >$ is well defined from the defition of CZO. Moreover, the convergence of the integral in the definiton follows from the regularity of the standard kernel and the special cancellation property due to the function g. One can easily check the defintion is independent of the choice of f_1. Now by a well-known result proved by Peetre, Spanne and Stein, the CZO extends to a bounded operator from L^∞ to BMO. Therefore, from the $BMO - H^1$ duality, "$T(1) \in BMO$" means there exists a constant $C > 0$ such that $< T(1), g >\leq C||g||_{H^1}$ for any $g \in C_c^\infty(R^n)$ with integral 0.

We need one more defintion. A continuous operator T from $S(R^n)$ to $S'(R^n)$ has the weak boundedness property if for any bounded subset B of $C_c^\infty(R^n)$, there exist a constant C dependent on B, such that for any ϕ_1, $\phi_2 \in B$, $x \in R^n$ and $R > 0$, we have $| < T\phi_1^{x,R}, \phi_2^{x,R} > | \leq CR^n$, where $\phi^{x,R}(y) = \phi((y - x)/R)$. Note that L^2 boundedness implies weak boundedness.

We now state the main theorem:

Theorem 3. *($T1$ theorem) Let T be a continuous operator from $S(R^n)$ to $S'(R^n)$, associated with a standard kernel. Then T can be extended to a bounded operator on $L^2(R^n)$ if and only if the following three conditions holds:*
i) $T1 \in BMO$,
ii) $T^*1 \in BMO$,
iii) T has the weak boundedness property.

The necessity of i), ii) and iii) follows from the discussion above. The miracle lies in the sufficiency. To prove sufficiency, the scheme is to decompose T as the sum of three operators L,M and $\tilde{T}$. The first two operators belong to the family of paraproducts. The L^2-boundedness of the third operator $\tilde{T}$ will be an application of Cotlar-Knapp-Stein lemma.

16.3 Paraproduct construction

The first part of the proof is to construct two CZOs, L and M, which are typical examples of a special class of bilinear operators called paraproduct and satisfy $L1 = T1$, $L^*1 = 0$, $M1 = 0$ and $M^*1 = T^*1$. Once we construct L, then M can be similarily constructed. We pick $\varphi \in C_c^\infty(R^n)$ be a radial function supported on the unit ball with integral 1. Denote $\varphi_t(x) = (1/t^n)\varphi(x/t)$, P_t to be the operator of convolution with φ_t and Q_t to be the operator $-tdP_t/dt$. It is easy to check Q_t is a operator of convolution with $\psi_t(x) = (1/t^n)\psi(x/t)$, where ψ is a radial atom supported on the unit ball. Let $\beta = T1 \in BMO$. We define the operator L formally by $Lf = \eta \int\limits_0^\infty Q_t[(Q_t\beta)(P_tf)]dt/t$, where η is some normalized constant to be chosen.

To show the operator L is a CZO, we need the following approximation lemma which can be easily deduced.

Lemma 4. *Let T_m be a bounded sequence of CZOs (i.e. the L^2-operator norms of T_m are uniformly bounded and the associated kernels estimates have the same constant C). If the associated kernels $K_m(x,y)$ converges uniformly on any compact set in Ω to $K(x,y)$ and the T_m converges weakly to an operator T, then T is a CZO with kernel K. Moreover, for any function $g \in C_c^\infty(R^n)$ with integral zero, $< T1, g >= \lim_{m\to+\infty} < T_m1, g >$.*

The lemma tells us how to approximate a CZO by a sequence of CZOs. We define $L_mf = \eta \int\limits_{1/m}^m Q_t[(Q_t\beta)(P_tf)]dt/t$ and the natural goal is to show:

Proposition 5. *The sequence L_m satisfies the hypothesis of the lemma.*

Given the explicit expression of L_m, we can directly verify the standard kernel condition. Denote $L_t(x,y) = \int \psi_t(x-z)Q_t\beta(z)\varphi_t(z-y)$ to be the kernel of $Q_t[(Q_t\beta)P_t]$. As we mentioned above Q_t is convolution with an atom, we know $Q_t\beta$ is uniformly by $C||\beta||_{BMO}$ for some constant C. Then the kernel of L_m and its derivative are bounded by $C||\beta||_{BMO}/|x-y|^n$ and $C||\beta||_{BMO}/|x-y|^{n+1}$, which satisfy the standard kernel condition.

The key tool to prove L^2-boundedness of L_m is the Carleson measure. For $f, g \in C_c^\infty(R^n)$, using Cauchy-Schwarz inequality, we have:

$$| < L_m f, g > | \leq (\int_{R^n} \int_{\frac{1}{m}}^{m} |Q_t g|^2 \frac{dxdt}{t})^{\frac{1}{2}} (\int_{R^n} \int_{\frac{1}{m}}^{m} |Q_t\beta|^2 |P_t f|^2 \frac{dxdt}{t})^{\frac{1}{2}} \qquad (3)$$

Using Plancherel theorem, the first factor can be estimated:

$$(\int_{R^n} \int_{\frac{1}{m}}^{m} |Q_t g|^2 \frac{dxdt}{t})^{\frac{1}{2}} \leq C||g||_2 \qquad (4)$$

For the second factor, we notice that $|Q_t\beta|^2 \frac{dxdt}{t}$ is a Carleson measure since $\beta \in BMO$. Therefore, using the Carleson measure estimate and the L^2-boundedness of Hardy-Littlewood maximal operator (for details of Carleson measure, see [2]), we have:

$$(\int_{R^n} \int_{\frac{1}{m}}^{m} |Q_t\beta|^2 |P_t f|^2 \frac{dxdt}{t})^{\frac{1}{2}} \leq C||f||_2 ||\beta||_{BMO} \qquad (5)$$

Now we know the L_m is a bounded sequence of CZOs and $< L_m f, g >$ converges. Therefore, L_m has a weak limit L, which is a CZO. Moreover, since $Q_t 1 = 0$, we have $L_m^* 1 = 0$ and therefore $L^* 1 = 0$. It remains to show $L1 = \beta$.

Lemma 6. *For all $g \in C_c^\infty(R^n)$ with integral 0, by choosing an appropriate η, we have*

$$\lim_{m\to\infty} < L_m 1, g > = < \beta, g >.$$

The proof of the lemma essentially uses the characterization of Hardy space by Riesz transform to transform the limit into a convergence of functions under L^1-norm. We omit the details here.

Now we have constructed two CZOs, L and M, satisfying $L1 = T1$, $L^*1 = 0$, $M1 = 0$ and $M^*1 = T^*1$. Therefore, the operator $\tilde{T} = T - L - M$ satifies $\tilde{T}1 = 0$, $\tilde{T}^*1 = 0$.

16.4 Almost orthogonality

We turn to the second part of the proof, which is to establish L^2-boundedness of $\tilde{T}$. The main ingredient to prove L^2-boundedness of $\tilde{T}$ is the following Cotlar-Knapp-Stein lemma (for a elegant proof, see [2]):

Lemma 7. *Let H be a Hilbert space, and T_j be a sequence of bounded operators on H. Let T_j^* be the adjoint of T_j. Suppose there exists a sequence $\omega : Z \to [0, +\infty)$ such that $\sum \sqrt{\omega(k)} = A < \infty$ and for all $(j,k) \in Z^2$, $||T_j^* T_k|| + ||T_j T_k^*|| \leq \omega(j-k)$. Then the sum $S_{M,N} = \sum_{j=-M}^{N} T_j$ converge strongly to a bounded operator T satisfying $||T|| \leq A$*

To use the lemma, we apply a Littlewood-Paley type decomposition to $\tilde{T}$. Let φ be as in the previous section. For $j \in Z$, denote S_j to be the convolution with φ_{2^j} and $\Delta_j = S_j - S_{j+1}$. Let $T_j = S_j \tilde{T} \Delta_j$, $T_j' = \Delta_j \tilde{T} S_j$ and $T_j'' = \Delta_j \tilde{T} \Delta_j$. Note $\sum_{-M}^{N} T_j + T_j' + T_j'' = S_{-M} \tilde{T} S_{-M} + S_N \tilde{T} S_N$ and S_j is an approximation to identity, so the partial sum converges weakly to $\tilde{T}$. We will show T_j satisfies CKS lemma and the other two can be deduced in the same way. The result is based on the following estimates of the kernel of T_j:

Lemma 8. *Denote $p_j(x) = 2^{-nj} p(x/2^j)$. The operator T_j is given by a kernel K_j such that:*

$$|K_j(x,y)| \leq C p_j(x-y); \tag{6}$$

$$\begin{aligned} &|K_j(x,y) - K_j(x',y)| + |K_j(y,x) - K_j(y,x')| \\ &\leq C Min(1, \frac{|x'-x|}{2^j})[p_j(x-y) + p_j(x'-y)]; \end{aligned} \tag{7}$$

$$\int K_j(x,y) dy = 0 \tag{8}$$

for all x;

$$\int K_j(x,y)dx = 0 \tag{9}$$

for all y.

By defintion, the kernel $K_j(x,y) =< \tilde{T}\phi^y_{2^j}, \varphi^x_{2^j} >$, where $\phi^y_{2^j} = \varphi^y_{2^j} - \varphi^y_{2^{j+1}}$. To prove (6), we have two cases. For $|x-y| \leq 10\ 2^j$, then (6) follows from the weak boundedness property of $\tilde{T}$. For $|x-y| \geq 10\ 2^j$, $\phi^y_{2^j}$ and $\varphi^x_{2^j}$ have disjoint supports. Using the fact that $\phi^y_{2^j}$ has integral 0, then (6) follows from the standard kernel estimate (2) associated to $\tilde{T}$.

For (7), we also have two cases. For $|x-x'| \geq 2^j$, then (7) follows from (6). For the other case, (7) can be reduced to the following gradient estimate of the kernel K_j:

$$|\Delta_x K_j(x,y)| + |\Delta_y K_j(x,y)| < C2^j p_j(x-y) \tag{10}$$

Note $\Delta_y K_j(x,y) = -2^j < \tilde{T}(\Delta\phi)^y_{2^j}, \varphi^x_{2^j} >$, so (10) follows from the proof of (6) applying to $\Delta\phi, \varphi$.

(8) and (9) follow from the fact that ϕ has integral 0 and $\tilde{T}^*1 = 0$.

Finally, using the estimates in the lemma, we can show T_js satisfy the hypothesis of the CKS lemma and thus complete the proof of the main theorem.

Proposition 9. *Let T_j be a sequence of operators with kernel K_j satisfying (6),(7),(8) and (9). Then there exists a constant $C > 0$ such that for all $(j,k) \in Z^2$,*

$$||T_j^* T_k|| + ||T_j T_k^*|| \leq C2^{-\delta|j-k|}. \tag{11}$$

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